frontiers in Applied Mathematics and Statistics

ORIGINAL RESEARCH
published: 30 March 2022
doi: 10.3389/fams.2022.847896

On the Global Positivity Solutions of Non-homogeneous Stochastic Differential Equations

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In this article, we treat the existence and uniqueness of strong solutions to the Cauchy problem of stochastic equations of the form $\mathrm{d}X_t = \alpha X_t \,\mathrm{d}t + \sigma X_t^{\gamma} \,\mathrm{d}B_t, X_0 = x > 0$. The construction does not require the drift and the diffusion coefficients to be Lipschitz continuous. Sufficient and necessary conditions for the existence of a global positive solution of non-homogeneous stochastic differential equations with a non-Lipschitzian diffusion coefficient are sought using probabilistic arguments. The special case $\gamma = 2$ and the general case, that is, $\gamma > 1$ are considered. A complete description of every possible behavior of the process X_t at the boundary points of the state interval is provided. For applications, the Cox-Ingersoll-Ross model is considered.

Keywords: geometric Brownian motion, Itô diffusion, Lipschitz continuous, scale function, speed measure

Edited by:
Ramoshweu Solomon Lebelo, Vaal University of Technology, South Africa

Reviewed by:
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Specialty section:
This article was submitted to Dynamical Systems, a section of the journal Frontiers in Applied Mathematics and Statistics

Received: *03 January 2022*
Accepted: *21 February 2022*
Published: *30 March 2022*

Citation:
Mhlanga FJ and Rundora L (2022) On the Global Positivity Solutions of Non-homogeneous Stochastic Differential Equations. Front. Appl. Math. Stat. 8:847896. doi: 10.3389/fams.2022.847896

1. INTRODUCTION

The theory of stochastic differential equations was developed by [1]. Stochastic differential equations are valuable tools for modeling systems and processes with stochastic disturbances in many fields of science and engineering. For the general theory of stochastic differential equations, one can refer to [2–5]. Several authors have discussed results concerning the existence and uniqueness of solutions of stochastic differential equations [2, 6, 7]. Mishura and Posashkova [8] provided a sufficient condition on coefficients which ensures almost surely positivity of the trajectories of the solution of the stochastic differential equation with non-homogeneous coefficients and non-Lipschitz diffusion. Appleby et al. [9] investigated highly non-linear stochastic differential equations with delays and showed that properties on the coefficients of stochastic differential equations that guarantee stability also guarantee positivity of solutions as long as the initial value is non-zero. Xu et al. [10] investigated the global positive solution of a stochastic differential equation, where they generalized the mean-reverting constant elasticity of variance process by replacing the constant parameters with the parameters modulated by a continuous-time, finite-state, Markov chain. Zhang [11] treated the properties of solutions to stochastic differential equations with Sobolev diffusion coefficients and singular drifts. Bae et al. [12] proved the existence of and uniqueness of solution to stochastic differential equations under weakened and Hölder conditions and a weakened linear growth condition. Conditions for positivity of solutions of fractional stochastic differential equations with coefficients that do not satisfy the linear growth Lipschitz continuous conditions were obtained by [13].

The aim of this article is to prove the existence of a global positive solution of stochastic differential equations of the form

$$\mathrm{d}X_t = \alpha X_t \,\mathrm{d}t + \sigma X_t^{\gamma} \,\mathrm{d}B_t, X_0 = x > 0, \tag{1}$$

where B_t is a standard Brownian motion, for different values of γ where α denotes the drift, σ denotes the volatility. $X = (X_t)_{t\geq 0}$ describes the underlying asset price. Such stochastic differential equations arise in modeling asset prices and interest rates on financial markets and it is crucial that X_t never becomes negative. Mao and Yuan [2] discussed the analytical properties when $\frac{1}{2} \leq \gamma \leq 1$ and showed that for a given initial value $X_0 = x > 0$, the solution of (1) remains positive with probability 1, namely, $X_t > 0$ for all $t \geq 0$ almost surely. The cases $\gamma = 0$ and $\gamma = 1$ give rise to the Ornstein-Uhlenbeck process and the Geometric Brownian motion, respectively, and this has been dealt with in the literature, see [4, 5, 14, 15]. When $\gamma > 1$, the diffusion coefficient of Equation (1) does not satisfy the linear growth condition, even though it is locally Lipschitz continuous. In view of this, it is not straightforward from the general theory of stochastic differential equations to obtain a unique global positive solution to Equation (1) that is defined for all $t \geq 0$. Nevertheless, there is a way to overcome such difficulties which we present in this article and we also provide detailed proofs that there is unique solutions to equations of the form (1). This article is an extension of the work done in [16] and [17] to non-homogeneous stochastic differential equations.

This article is structured as follows. In Section 2, we consider the existence of a positive global solution for non-homogeneous stochastic differential equations with non-Lipschitz coefficients. In particular, we treat the case $\gamma = 2$ and prove that if $\alpha \geq 0$ and $x \geq 0$ is arbitrary, then a unique strong solution of Equation (1) exists. Section 3 deals with the existence and uniqueness of a positive global solution to non-homogeneous stochastic differential equation (1). In particular, we consider the general case, that is, $\gamma > 1$. We provide a detailed proof of the existence of a unique solution to Equation (1). In Section 4, we investigate the behavior of the underlying process X_t at the boundaries of the state space $(0, \infty)$. The main tool used are simple probabilistic arguments. We only require the coefficients of our model to be continuous in the usual sense. In Section 5, we provide a brief conclusion.

2. EXISTENCE OF POSITIVE GLOBAL SOLUTIONS: $\gamma = 2$

We want to prove that a unique global positive solution to Equation (1) exists and investigate its properties. We notice that if $X_0 = x = 0$, then by strong uniqueness we have $X_t = 0$ for all $t \geq 0$. In addition, if a solution X_t exists for all $t < \tau(\omega) \leq \infty$ for some $x > 0$ and $X_T = 0$ for some $T = T(\omega) < \tau(\omega)$, then by the Strong Markov property we have$X_t = 0$ for all $t \in [T(\omega), \tau(\omega)]$. In particular, $x \geq 0$ implies that $X_t \geq 0$.

We call $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ a stochastic basis if $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $(\mathcal{F}_t)$ is a right continuous filtration on Ω augmented by the $\mathbb{P}$-null sets. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

We consider a stochastic differential equation of the form

$$\mathrm{d}X_t = \alpha(X_t) \,\mathrm{d}t + \sigma(X_t) \,\mathrm{d}B_t, \tag{2}$$

where the coefficients $\alpha : \mathbf{R} \to \mathbf{R}$ and $\sigma : \mathbf{R} \to \mathbf{R}$ are both Borel measurable functions. By the definition of stochastic differential, Equation (2) is equivalent to the stochastic integral equation:

$$X_t = x + \int_0^t \alpha(X_s) \,\mathrm{d}s + \int_0^t \sigma(X_s) \,\mathrm{d}B_s. \tag{3}$$

Definition 2.1. *[2, p. 48] An* $\mathbf{R}$*-valued stochastic process* $\{X_t\}_{t\in[0,T]}$ *is called a solution of Equation (2) if it has the following properties:*

1. $\{X_t\}$ *is continuous and* $\mathcal{F}_t$*-adapted.*
2. $\int_0^t | \alpha(X_s) | \,\mathrm{d}s < \infty$ *and* $\int_0^t \sigma^2(X_s)\mathrm{d}s < \infty$.
3. *Equation (3) holds for every* $t \in [0, T]$ *with probability* 1.

Definition 2.2. *[18, p. 167] A solution* (X, B) *of Equation (2) defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ *is said to be a strong solution if* X *is adapted to the filtration* $\{\mathcal{F}_t^B\}$*, that is, the filtration of* $B = (B_t)_{t\geq 0}$ *completed with respect to* $\mathbb{P}$.

Definition 2.3. *[7, p. 300] A weak solution is a triple* $((\Omega, \mathcal{F}, \mathbb{P}), (B, X), (\mathcal{F}_t))$ *where* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $\{\mathcal{F}_t\}$ *is a filtration of sub-*σ*-fields of* $\mathcal{F}$ *satisfying the usual conditions,* X *is a continuous, adapted* $\mathbf{R}$*-valued process,* B *is the standard Brownian motion such that Equation (3) is satisfied.*

Remark 2.1.

1. Definition 2.2 says that if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, the Brownian motion B_t and the coefficients $\alpha(x)$ and $\sigma(x)$ are all given in advance, and then the solution X_t is constructed, such a solution is called a strong solution.
2. Definition 2.3 says that if we are only given the coefficients $\alpha(x)$ and $\sigma(x)$, and we are allowed to construct a suitable probability space, a filtration and find a solution to the Equation (2), then such a solution is called a weak solution.

A solution $\{X_t\}_{t\in[0,T]}$ is said to be unique if any other solution $\{\overline{X}_t\}$ is indistinguishable from $\{X_t\}$, that is

$$\mathbb{P}(X_t = \overline{X}_t \quad \forall \ 0 \leq t \leq T) = 1.$$

Following [19], we impose the following hypothesis:

(H) The drift coefficient is globally Lipschitz, that is, for all $x, y \in \mathbf{R}$,

$$| \alpha(x) - \alpha(y) | \leq K | x - y | \tag{4}$$

where K is a fixed constant, while the diffusion coefficient is globally Hölder continuous, that is, for all $x, y \in \mathbf{R}$,

$$| \sigma(x) - \sigma(y) | \leq h(| x - y |) \tag{5}$$

where $h:[0,\infty) \rightarrow [0,\infty)$ is a strictly increasing function with $h(0)=0$ and the integral

$$\int_0^{\varepsilon} \frac{du}{h^2(u)} = \infty, \quad \forall \varepsilon > 0.$$

It is known [see [19]] that under the hypothesis (H), the strong uniqueness solution holds for the stochastic differential equation (2).

For the case $\gamma = 2$, Equation (1) takes the form

$$\mathrm{d}X_t \; = \alpha X_t \,\mathrm{d}t + \sigma X_t^2 \,\mathrm{d}B_t. \tag{6}$$

If we let $Y_t = \ln \mid X_t \mid$ then an application of Itô's formula yields

$$\mathrm{d}Y_t \; = (\alpha - \frac{1}{2}\sigma^2 X_t^2)\,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t \tag{7}$$

which is equivalent to

$$\ln \mid X_t \mid = \ln \mid x \mid + \alpha t - \frac{1}{2}\sigma^2 \int_0^t X_s^2 \,\mathrm{d}s + \sigma \int_0^t X_s \,\mathrm{d}B_s. \tag{8}$$

This solution presents a challenge as the coefficients in Equation (8) do not satisfy the linear growth and Lipschitz conditions. However, there is a way to go around this. In the next result we prove the existence of a global solution to Equation (6) following arguments presented in [20].

Theorem 2.1. *Suppose $\alpha \geq 0$ and $x \geq 0$ is arbitrary, then the stochastic differential equation of the form (6) has a unique, strong solution X_t defined for all $t \geq 0$.*

Proof: The result is proved by a truncation procedure. For each $n \geq 1$, we set $\alpha = \alpha_n(x)$ and the truncation function

$$\sigma_n(x) = \begin{cases} \sigma x^2 & \text{if} \mid x \mid \leq n, \\ \sigma n^2 & \text{if} \mid x \mid > n \end{cases} .$$

Then, $\alpha_n(x)$ and $\sigma_n(x)$ satisfy the hypothesis (H). Hence, there is a unique solution $X_t = X_t^{(n)}$ defined for all t to the equation

$$X_t^{(n)} \; = x + \int_0^t \alpha_n(X_s^{(n)})\,\mathrm{d}s + \int_0^t \sigma_n(X_s^{(n)})\,\mathrm{d}B_s. \tag{9}$$

Define the stopping time

$$\tau_n = \inf\{t > 0; \mid X_t^{(n)} \mid \geq n\}, \quad n = 1, 2, ... \tag{10}$$

Then, by strong uniqueness we have

$$X_t^{(n)}(\omega) = X_t^{(n+1)}(\omega) \;\text{ for all }\; t \leq \tau_n \quad a.s. \tag{11}$$

Therefore,

$$\tau_n \; = \inf\{t > 0; \mid X_t^{(n)} \mid \geq n\} < \inf\{t > 0; \mid X_t^{(n+1)} \mid \geq n+1\} = \tau_{n+1}. \tag{12}$$

Hence, $\{\tau_n\}$ is an increasing sequence of stopping times. Put

$$\tau(\omega) = \lim_{n\to\infty} \tau_n(\omega) \leq \infty.$$

Then, for $t < \tau(\omega)$, a process X_t can be defined by setting

$$X_t(\omega) = X_t^{(n)}(\omega), \quad \text{if} \quad t < \tau_n(\omega). \tag{13}$$

It is clear that if $t < \tau(\omega)$ then one can easily show that $t < \tau_n(\omega)$ for some n. Therefore, by (11), this defines $X_t(\omega)$ uniquely. Hence, we have

$$X_t = x + \int_0^t \alpha X_s \,\mathrm{d}s + \int_0^t \sigma X_s^2 \,\mathrm{d}B_s \quad \text{for} \quad t < \tau(\omega). \tag{14}$$

3. EXISTENCE AND UNIQUENESS OF POSITIVE GLOBAL SOLUTION

In this section, we provide a detailed proof that there is a unique positive global solution to Equation (1). In particular, we focus on the case $\gamma > 1$. To establish the existence of a unique positive global solution, we need the following result.

Lemma 3.1. *[3, p. 57] The coefficients of Equation (1) satisfy the local Lipschitz condition for given initial condition $X_0 = x > 0$, that is, for every integer $k > 1$, there exists a positive constant L_k and $x, y \in [0,k]$ such that*

$$\mid \alpha x - \alpha y \mid^2 + \mid \sigma x^\gamma - \sigma y^\gamma \mid^2 \leq L_k \mid x - y \mid^2 . \tag{15}$$

Therefore, there exists a unique local solution to Equation (1).

We now state our result in the following theorem.

Theorem 3.1. *For any given initial value $X_0 = x > 0$, α and $\sigma > 0$ there exists a unique positive global solution X_t to Equation (1) on $t \geq 0$ for $\gamma > 1$.*

Proof: It is clear that the coefficients of Equation (1) are locally Lipschitz continuous. Therefore, for any given initial value $X_0 = x > 0$, there is a unique local solution X_t, $t \in [0, \tau(\omega)]$ of Equation (1) where $\tau(\omega)$ is the explosion time. To prove that the solution is global, it suffices to show that $\tau(\omega) = \infty$ almost surely. We prove this by contradiction. If $\tau(\omega) \neq \infty$, then we can find a pair of positive constants ϵ and T such that

$$\mathbb{P}(\tau(\omega) \leq T) > \epsilon. \tag{16}$$

For each integer $n > 1$, we define a stopping time

$$\tau_n = \inf\{t \geq 0 : \mid X_t \mid \geq n\}. \tag{17}$$

Since $\tau_n \rightarrow \tau(\omega)$ almost surely, we can find a sufficiently large n_0 for which

$$\mathbb{P}(\tau_n \leq T) > \frac{\epsilon}{2}, \quad \text{for all } \; n \geq n_0. \tag{18}$$

For $\theta, \beta > 0$, we define a function $V \in C^2$ as

$$V(X) := \theta\sqrt{X} + \beta X^{-2}, \tag{19}$$

which is continuously twice differentiable in X. We observe that $V(X) \to +\infty$ as $X \to +\infty$ or $X \to 0$. For any $0 < t < T$, an application of Itô formula gives

$$\mathrm{d}V(X_t) = \mathcal{L}V(X_t)\,\mathrm{d}t + \sigma X^\gamma \left(\frac{1}{2}\theta X_t^{-\frac{1}{2}} - 2\beta X_t^{-3}\right)\mathrm{d}B_t, \tag{20}$$

where

$$\begin{aligned}\mathcal{L}V(X_t) &= \alpha X_t \left(\frac{1}{2}\theta X_t^{-\frac{1}{2}} - 2\beta X_t^{-3}\right) \\ &\quad + \frac{1}{2}\sigma^2 X_t^{2\gamma}\left(-\frac{1}{4}\theta X_t^{-\frac{2}{3}} + 6\beta X_t^{-4}\right).\end{aligned} \tag{21}$$

By boundedness of polynomials, there exists a constant K such that

$$\alpha X_t \left(\frac{1}{2}\theta X_t^{-\frac{1}{2}} - 2\beta X_t^{-3}\right) + \frac{1}{2}\sigma^2 X_t^{2\gamma}\left(-\frac{1}{4}\theta X_t^{-\frac{2}{3}} + 6\beta X_t^{-4}\right) \le K. \tag{22}$$

Therefore, for any $t \in [0, T]$

$$\begin{aligned}\mathbf{E}[V(X_{t\wedge\tau_n})] &= V(x) + \mathbf{E}\left[\int_0^{t\wedge\tau_n} \mathcal{L}V(X_s)\,\mathrm{d}s\right] \le V(x) + KT \\ &\quad + K\mathbf{E}\left[\int_0^t \mathbf{E}[V(X_{s\wedge\tau_n})]\,\mathrm{d}s\right].\end{aligned} \tag{23}$$

The application of the Grownwall inequality yields

$$\mathbf{E}[V(X_{T\wedge\tau_n})] \le [V(x) + KT]e^{KT} \tag{24}$$

which is equivalent to

$$\mathbf{E}[V(X_{\tau_n})\mathbf{1}_{\{\tau_n \le T\}}] \le [V(x) + KT]e^{KT}. \tag{25}$$

On the other hand, we define

$$M_n = \inf\{V(X_t) \mid X_t > n,\ t \in [0, T]\}. \tag{26}$$

As $n \to +\infty$, $M_n \to +\infty$. It now follows from (18) and (26) that

$$[V(x) + KT]e^{KT} \ge M_n\mathbb{P}(\{\tau_n \le T\}) \ge \frac{1}{2}\epsilon M_n. \tag{27}$$

Letting $n \to +\infty$ yields a contradiction, so we must have $\tau(\omega) = \infty$ almost surely. Therefore, there exists a unique positive global solution X_t to Equation (1) for all $t \ge 0$.

4. ANALYSIS OF THE SOLUTION AT THE BOUNDARIES OF THE STATE SPACE

We now investigate the behavior of the underlying process X_t at the boundaries of the state space $(0, \infty)$ using probability arguments. X_t is the solution of the stochastic differential equation (1), where X_t is defined on the state space $(0, \infty)$, that is, the whole positive real line.

We first consider the Itô diffusion of the form

$$\mathrm{d}X_t = \alpha(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x, \tag{28}$$

where $\alpha : \mathbf{R} \to \mathbf{R}$ and $\sigma : \mathbf{R} \to \mathbf{R}$ are functions satisfying the hypothesis (H). Note that here we do not have the time argument. We assume that the state space of X_t is a finite or infinite interval. Such a process is a continuous Markov process and under weak regularity conditions the drift coefficient $\alpha(x)$ and the diffusion coefficient $\sigma(x)$ are characterized, respectively, by

$$\alpha(x) = \lim_{h\downarrow 0} h^{-1}\mathbf{E}[X_h - x] \tag{29}$$

and

$$\sigma^2(x) = \lim_{h\downarrow 0} h^{-1}\mathbf{E}[(X_h - x)^2] = \lim_{h\downarrow 0} h^{-1}\mathrm{Var}(X_h). \tag{30}$$

For details about these conditions as well as the foregoing, see [21]. The above conditions can conveniently be weakened to give the following three conditions.

$$h^{-1}\mathbf{E}[(X_h - x)\mathbf{1}_{\{|X_h - x| \le 1\}}] \to \alpha(x), \tag{31}$$

$$h^{-1}\mathbf{E}[(X_h - x)^2\mathbf{1}_{\{|X_h - x| \le 1\}}] \to \sigma^2(x), \tag{32}$$

and

$$h^{-1}\mathbb{P}(|X_h - x| > \varepsilon) \to 0 \ \forall\ \varepsilon > 0, \tag{33}$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. These conditions enable us to perform the analysis of (28) without assuming the Lipschitz conditions to the coefficients. We will, however, assume that $\alpha(x)$ and $\sigma(x)$ are continuous.

Fix $q \in \mathbf{R}$ and define the scale function u by

$$u(x) = \int_q^x \exp\left(-\int_q^t \frac{2\alpha(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\mathrm{d}t, \quad u(q) = 0. \tag{34}$$

The function u has a continuous, strictly positive derivative and u'' exists almost everywhere and satisfies

$$u''(x) = -\frac{2\alpha(x)}{\sigma^2(x)}u'(x). \tag{35}$$

We also introduce the speed measure

$$m(dx) = \frac{2}{u'(x)\sigma^2(x)}\,\mathrm{d}x. \tag{36}$$

Now, let $p(t, x, y)$ be the transition density of X_t. Then, the Kolmogorov backward equation is given by

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2} + \alpha(x)\frac{\partial p}{\partial x}. \tag{37}$$

At $t = 0$, $p(0, x, y) = \delta(x - y)$, where $\delta(\cdot)$ is Dirac's delta function.

Let $[a, b]$ be a fixed interval and start the process at $X_0 = x \in (a, b)$. We want to find the probability $p_+(x)$ that the process X_t hits b before it hits a. By the Markov property, we have

$$p_+(x) = \mathbf{E}[p_+(X_s)] + O\left(\mathbb{P}(|X_s - x| > \varepsilon)\right).$$

It follows from Equation (33) that

$$s^{-1}\left(\mathbb{P}(|X_s - x| > \varepsilon)\right) \to 0,$$

when $s \downarrow 0$ if $a + \varepsilon < x < b - \varepsilon$. Using the Itô's formula and Equation (37), we can show that $p_+(x)$ satisfies the Kolmogorov's backward equation

$$\frac{1}{2}\sigma^2(x)p_+''(x) + \alpha(x)p_+'(x) = 0, \tag{38}$$

for $x \in (a, b)$ with the boundary conditions $p_+(a) = 0$ and $p_+(b) = 1$. The explicit solution to Equation (38) is

$$p_+(x) = A\int_q^x \exp\left(-\int_q^t \frac{2\alpha(z)}{\sigma^2(z)}\,\mathrm{d}z\right)\mathrm{d}t + B. \tag{39}$$

We can write Equation (39) in the form

$$p_+(x) = Au(x) + B, \tag{40}$$

where $u(x)$ is of the form Equation (34) for a fixed $q \in (a, b)$, with A and B constants. Now, an application of boundary conditions $p_+(a) = 0$ and $p_+(b) = 1$ gives:

$$A = \frac{1}{u(b) - u(a)} \quad \text{and} \quad B = \frac{-u(a)}{u(b) - u(a)},$$

so that

$$p_+(x) = \frac{u(x) - u(a)}{u(b) - u(a)}. \tag{41}$$

Equations (34) and (41) will be important when applied to our specific problem.

Following similar arguments, we define

$$e(x) = \mathbf{E}[T_{ab}], \tag{42}$$

where $T_{ab} = \inf\{t > 0 : X_t \notin (a, b)\}$. An application of the Markov property gives

$$e(x) = s + \mathbf{E}[e(X_s)] + O(\mathbb{P}(|X_s - x| > \varepsilon)).$$

Dividing by s and letting s tend to 0 and an application of the Itô formula gives

$$\frac{1}{2}\sigma^2(x)e''(x) + \alpha(x)e'(x) = -1. \tag{43}$$

This equation can be solved by the standard Green function techniques as follows. The corresponding homogeneous equation is

$$e''(x) + \frac{2\alpha(x)}{\sigma^2(x)}e'(x) = 0$$

and its solution is

$$e(x) = \frac{u(x) - u(a)}{u(b) - u(a)}, \tag{44}$$

with boundary conditions $e(a) = 0$ and $e(b) = 1$ where $u(x)$ is defined in Equation (34). The Green function, $G_{(a,b)}(x, y)$, is calculated as

$$G_{(a,b)}(x, y) = \begin{cases} \frac{1}{W} \cdot e_1(x)e_2(y) & \text{if } x \leq y, \\ \frac{1}{W} \cdot e_1(y)e_2(x) & \text{if } x \geq y, \end{cases} \tag{45}$$

where e_1 and e_2 take the form of Equation (44) and W is the Wronskian given by

$$W = \frac{u'(x)}{u(b) - u(a)}.$$

Therefore, the solution to Equation (43) is given by

$$e(x) = \int_a^b G_{(a,b)}(x, y)m(\mathrm{d}y),$$

where $G_{(a,b)}(x, y)$ is given by

$$G_{(a,b)}(x, y) = \begin{cases} \frac{2(u(x)-u(a))(u(b)-u(y))}{u(b)-u(a)} & \text{if} \quad x \leq y, \\ \\ \frac{2(u(y)-u(a))(u(b)-u(x))}{u(b)-u(a)} & \text{if} \quad x > y, \end{cases} \tag{46}$$

and $m(dy)$ is given by Equation (36).

We now consider Equation (1). We note that the diffusion coefficient $\sigma(x) = \sigma x^\gamma$ in Equation (1) is defined only on $(0, \infty)$, that is, the state space of the process is made up of the positive reals. The process X_t in Equation (1) is a diffusion process, and the coefficients σ and α are continuous on $(0, \infty)$. Following the arguments in [17], we investigate the behavior of Equation (1) at the boundaries of the state space. It is of interest whether or not the boundary points 0 and/or ∞ can be reached by the process X_t in a finite time.

A boundary point is said to be accessible if it can be reached in finite time with positive probability. Otherwise it is inaccessible [17]. The accessible boundary points are of two different types, namely, the exit and regular boundary points. For the exit boundary, the process is absorbed after the boundary is reached while the regular boundary point is imposed on a standard Brownian motion and could either be absorbed or reflected once the boundary is reached. The inaccessible boundaries are also of two types, namely, the entrance and natural boundary points. The boundary is said to be of entrance type if it is possible to start the process at infinity and then reach the interior of the state interval, otherwise it is called natural.

Let $[a, b]$ be a fixed interval and the process X_t starts in $X_0 = x \in (a, b)$. Let α and σ be continuous on a state interval whose interior is (c, d). We note that we may have $c = -\infty$ and/or $d = +\infty$. It is assumed that $\sigma^2(x) > 0$ on (c, d). Further, let $(\widetilde{c}, \widetilde{d}) = (u(c), u(d))$, where u is the scale function given by Equation (34).

Definition 4.1. *A natural upper boundary point d is said to be attracting if there is a positive probability that X_t shall converge to d as $t \to \infty$.*

The following classification theorem, taken from [17], will be the framework of the analysis of Equation (1).

Theorem 4.1. *Let u be the scale function given by Equation (34) and $m(\mathrm{d}y)$ be the speed measure given by Equation (36). Let b be a point in the interior of the state space (c, d). Then, the following statements hold.*

1. *A necessary and sufficient condition for d to be accessible is that $u(d) < \infty$ and $\int_b^d (u(d) - u(y))m(\mathrm{d}y) < \infty$.*
2. *An accessible boundary point d is regular if and only if $\int_b^d m(\mathrm{d}y) < \infty$. Otherwise it is exit boundary.*
3. *An inaccessible boundary point d is natural if and only if $\int_b^d u(y)m(\mathrm{d}y) = \infty$.*
4. *A natural boundary point d is attracting if and only if $u(d) < \infty$ and at the same time $\int_b^d m(\mathrm{d}y) = \infty$.*

We are now in a position of analyzing the non-homogeneous stochastic differential equation (1), repeated here for convenience,

$$\mathrm{d}X_t = \alpha X_t\,\mathrm{d}t + \sigma X_t^{\gamma}\,\mathrm{d}B_t,\ \ \gamma > 1,\ \ X_0 = x > 0. \tag{47}$$

This is a diffusion process with $\alpha(x) = \alpha x$, $\sigma(x) = \sigma x^{\gamma}$ and natural state interval $c = 0$ to $d = \infty$. Let b be a point in the interior of this state interval. From Equation (34) and (36), we calculate the scale function and speed measure, respectively, corresponding to Equation (47) to be

$$u(x) = \frac{\sigma^2}{2\alpha}\left(b^{2\gamma-1} - x^{2\gamma-1}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}(b^{2-2\gamma} - x^{2-2\gamma})\right)\right),$$

and

$$m(dy) = \frac{1}{\sigma^2 y^{2\gamma}}\exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)}(b^{2-2\gamma} - y^{2-2\gamma})\right)\mathrm{d}y. \tag{48}$$

It remains only to classify our boundary points on the basis of these results. We note that

$$u(d) = \frac{\sigma^2}{2\alpha}\left(b^{2\gamma-1} - d^{2\gamma-1}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}(b^{2-2\gamma} - d^{2-2\gamma})\right)\right).$$

Since $d = \infty$ in our state space $(0, \infty)$, we use a limiting argument:

$$\lim_{d\to\infty} u(d) = \frac{\sigma^2 b^{2\gamma-1}}{2\alpha},$$

provided $0 < \gamma < \frac{1}{2}$. Now, since b is a finite fixed point in $(0, \infty)$, the limit is finite. We also need to investigate the integral

$$\begin{aligned}\int_b^d (u(d) - u(y))m(\mathrm{d}y) = \\ \int_b^d \frac{1}{2\alpha y}\,\mathrm{d}y - \frac{1}{2\alpha}\exp\left(\frac{-\alpha}{\sigma^2(1-\gamma)}d^{2-2\gamma}\right)d^{2\gamma-1} \\ \times \int_b^d \frac{1}{y^{2\gamma}}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}y^{2-2\gamma}\right)\mathrm{d}y.\end{aligned} \tag{49}$$

We observe that as $d \to \infty$ the term $\exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)}d^{2-2\gamma}\right)$ approaches 0 provided $0 < \gamma < 1$. So in this case we remain with

$$\int_b^d (u(d) - u(y))m(\mathrm{d}y) \approx \int_b^d \frac{1}{2\alpha y}\,\mathrm{d}y \longrightarrow \infty \ \ as \ \ d \to \infty.$$

We therefore, according to Theorem 4.1, conclude that the upper boundary point $d = \infty$ is inaccessible if $0 < \gamma < 1$. Now for the lower boundary point 0 we have

$$u(0) = \frac{\sigma^2 b^{2\gamma-1}}{2\alpha} < \infty,$$

since b is a finite fixed point in the state space $(0, \infty)$. We also have

$$\int_0^b (u(0) - u(y))m(\mathrm{d}y) = \int_0^b \frac{1}{2\alpha y}\,\mathrm{d}y \to \infty,\ \ \text{as } y \to 0.$$

This shows that the boundary point 0 is inaccessible for all $\gamma \neq 1$. It remains to establish whether our boundary points are natural or not. Theorem 4.1 says the boundary point d is natural if and only if

$$\int_b^d u(y)m(\mathrm{d}y) = \infty.$$

Now,

$$\begin{aligned}&\int_b^d u(y)m(\mathrm{d}y) = \\ &\int_b^d \left(\frac{b^{2\gamma-1}}{2\alpha y^{2\gamma}}\exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)}(b^{2-2\gamma} - y^{2-2\gamma})\right) - \frac{1}{2\alpha y}\right)\mathrm{d}y \\ &= \frac{b^{2\gamma-1}\exp\left(\frac{-\alpha}{\sigma^2(1-\gamma)}b^{2-2\gamma}\right)}{2\alpha}\int_b^d \frac{1}{y^{2\gamma}}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}y^{2-2\gamma}\right)\mathrm{d}y \\ &- \int_b^d \frac{1}{2\alpha y}\,\mathrm{d}y.\end{aligned}$$

We observe that for $0 < \gamma < 1$ the integral

$$\int_b^d \frac{1}{y^{2\gamma}}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}y^{2-2\gamma}\right)\mathrm{d}y$$

explodes to infinity very fast as $d \rightarrow \infty$. Although the second integral,

$$\int_b^d \frac{1}{2\alpha y}\,\mathrm{d}y,$$

also tends to infinity as $d \to \infty$, the whole integral $\int_b^d u(y)m(\mathrm{d}y)$ tends to infinity as $d \to \infty$ because the second integral goes to infinity very slowly as compared to the first one. Hence,

$$\int_b^d u(y)m(\mathrm{d}y) = \infty,$$

provided $0 < \gamma < 1$. This tells us that the boundary point $d = \infty$ is natural if $0 < \gamma < 1$.
Using similar arguments, we can show that

$$\int_0^b u(y)m(\mathrm{d}y) = \infty.$$

Therefore, for $0 < \gamma < 1$, the boundary point 0 is natural. Next, we investigate if our natural boundary points are attracting. According to Theorem 4.1 the boundary point d is attracting if and only if $u(d) < \infty$ and at the same time $\int_b^d m(\mathrm{d}y) = \infty$. Now we have already seen that $u(d) < \infty$ if $d = 0$ and/or $d = \infty$ provided $0 < \gamma < 1$. Further

$$\int_b^d m(\mathrm{d}y) = \frac{1}{\sigma^2}\exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)}b^{2-2\gamma}\right) \times \int_b^d \frac{1}{y^{2\gamma}}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}y^{2-2\gamma}\right)\mathrm{d}y,$$

which, for the reason given before, explodes to infinity as $d \to \infty$ for all $0 < \gamma < 1$. Therefore,

$$\int_b^\infty m(\mathrm{d}y) = \infty,$$

for $0 < \gamma < 1$. Hence, the upper boundary point $d = \infty$ is attracting for $0 < \gamma < 1$. Similarly the lower boundary point 0 is shown to be attracting.

Now, we have established that both boundary points are attracting when $0 < \gamma < 1$. In this case we will show that, by Equation (41), our process will converge to ∞ with probability $p_+(x)$, where $x = X_0 \in (0, \infty)$. It turns out that

$$p_+(x) = \frac{\int_0^x \exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)}y^{2-2\gamma}\right)\mathrm{d}y}{\int_0^\infty \exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)}y^{2-2\gamma}\right)\mathrm{d}y}, \quad 0 < \gamma < 1. \tag{50}$$

Evaluating the integrals yields

$$p_+(x) = \lim_{y\to\infty}\left(\frac{x}{y}\right)^{2\gamma-1}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}(y^{2-2\gamma} - x^{2-2\gamma})\right) = 0,$$

for $\frac{1}{2} < \gamma < 1$. This shows that although the upper boundary $d = \infty$ is attracting for $0 < \gamma < 1$, the process X_t will not converge to ∞ if $\frac{1}{2} < \gamma < 1$. Furthermore, the process X_t converges to 0 with probability $1 - p_+(x)$ which turns out to be 1 in this case. Thus it is certain that X_t will converge to 0 when $\frac{1}{2} < \gamma < 1$. We observe that if $0 < \gamma < \frac{1}{2}$ we have a problem since, in this case, it is not possible to proceed using a probabilistic argument. Our analysis is not complete if we do not consider the case $\gamma > 1$. We now proceed to make this analysis. As seen earlier

$$u(d) = \frac{\sigma^2}{2\alpha}\left(b^{2\gamma-1} - d^{2\gamma-1}\exp\left(\frac{\alpha}{\sigma^2(1-\gamma)}(b^{2-2\gamma} - d^{2-2\gamma})\right)\right).$$

If $\gamma > 1$, for example, if $\gamma = 2$, we have

$$u(d) \longrightarrow -\infty \quad as \quad d \to \infty,$$

since α, σ and b are fixed positive numbers. Therefore, we have

$$\lim_{d\to\infty} u(d) < \infty \quad \forall \quad \gamma > 1.$$

Observe also that for such γ we have that

$$u(0) = \frac{\sigma^2 b^{2\gamma-1}}{2\alpha} < \infty,$$

effectively. Now, we consider again Equation (49). If $\gamma > 1$, for instance, $\gamma = 2$, we have

$$\int_b^d (u(d) - u(y))m(\mathrm{d}y) = \int_b^d \frac{1}{2\alpha y}\,\mathrm{d}y - \frac{d^3}{2\alpha}e^{\frac{\alpha}{\sigma^2 d^2}}\int_b^d \frac{1}{y^4}e^{-\frac{\alpha}{\sigma^2 y^2}}\,\mathrm{d}y.$$

We immediately observe that as $d \to \infty$,
$\int_b^d (u(d) - u(y))m(\mathrm{d}y) \to -\infty$ since the integral $\int_b^d \frac{1}{2\alpha y}\,\mathrm{d}y$ goes to ∞ very slowly. Therefore, effectively we have

$$\int_b^d (u(d) - u(y))m(\mathrm{d}y) < \infty,$$

for the boundary point $d = \infty$ and whenever $\gamma > 1$. This, together with $u(d) < \infty$ for $d = \infty$ and $\gamma > 1$, shows that the upper boundary point $d = \infty$ is always accessible whenever $\gamma > 1$. However, it is clear that if $d = 0$,

$$\int_0^b (u(0) - u(y))m(\mathrm{d}y) = \int_0^b \frac{1}{2\alpha y}\,\mathrm{d}y = \infty,$$

which shows that the lower boundary point 0 is always not accessible for $\gamma > 1$. In fact, the boundary point 0 is always inaccessible for all $\gamma \neq 1$ as also shown earlier. So from definition we have seen that the upper boundary point ∞ can be reached in finite time with positive probability provided $\gamma > 1$.

Finally we want to classify the accessible boundary point ∞, that is. is it regular or exit? From Theorem 4.1 it is regular if and only if $\int_b^d m(\mathrm{d}y) < \infty$. Now, as obtained earlier on

$$\int_b^d m(\mathrm{d}y) = \frac{1}{\sigma^2} \exp\left(-\frac{\alpha}{\sigma^2(1-\gamma)} b^{2-2\gamma}\right) \times \int_b^d \frac{1}{y^{2\gamma}} \exp\left(\frac{\alpha}{\sigma^2(1-\gamma)} y^{2-2\gamma}\right) \mathrm{d}y.$$

If $\gamma > 1$, the integral is always less than ∞ because of the negative exponent since α, σ are fixed positive numbers. So we have

$$\int_b^d m(\mathrm{d}y) < \infty \quad \forall \gamma > 1$$

since in this case the exponent is always negative. Hence for $\gamma > 1$, the accessible upper boundary point ∞ is always regular, i.e., apart from absorption and reflection there are also other possibilities after the boundary point is reached.
We, therefore, have the following result.

Theorem 4.2. *Let $x \in (0, \infty)$ with $\alpha \in \boldsymbol{R}$ arbitrary. Then, the stochastic differential equation (1) has a unique strong global solution $X_t : t \geq 0$. The solution has the following properties:*

1. *$x = 0$ implies that $X_t = 0$ for all $t > 0$ and $x \geq 0$ implies $X_t > 0$ for all $t \geq 0$.*
2. *If $\frac{1}{2} < \gamma < 1$, then $\lim_{t\to\infty} X_t = 0$ with probability $1 - p_+(x)$ where $p_+(x)$ is given by Equation (50).*
3. *If $\gamma > 1$, then $\lim_{t\to\infty} X_t = \infty$ with positive probability.*
4. *If $\gamma = 1$, we have the usual Geometric Brownian motion whereas if $\gamma = 0$, we have the Ornstein-Uhlenbeck process.*

In mathematical finance, our result is of particular interest for the Cox-Ingersoll-Roll (CIR) model which describes the stochastic evolution of interest rates $(r_t)_{t\geq 0}$ by the stochastic differential equation

$$\mathrm{d}r_t = \alpha(\mu - r)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}B_t, \quad t \geq 0,$$

with $r_0 \geq 0$ and $\alpha\mu \geq \frac{1}{2}\sigma^2$ where α, μ and σ denote real constants.

5. CONCLUDING REMARKS

In this article, we proved the existence of global positive solutions to non-homogeneous stochastic differential equations whose diffusion coefficient is non-Lispchitz. We relied on both the classical sense and probabilistic arguments. We provided detailed proofs in both cases. The probability arguments save as an alternative method of dealing with non-homogeneous stochastic differential equations where classical methods cannot be applied. Using the scale function and the speed of measure, we provided a complete classification of boundary types and boundary behavior of Equation (1). The results of this article can be applied to Cox-Ingersoll-Ross model. In addition, the positivity of solutions is important to other non-linear models that arise in sciences and engineering.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

FM contributed to the conceptualization of the idea for research, development of the methodology, analysis of the model and writing up of the final article. LR conducted the primary research, identified the problem, and discussed the results. Both authors proof read the manuscript and approved the submitted version.

FUNDING

This work was supported by a Newton Fund's Operational Development Assistance grant, ID 32, under the SA-UK partnership. The grant is funded by the UK Department for Business, Energy and Industrial Strategy and Department of Higher Education and Training (DHET) and delivered by the British Council. For further information, please visit www.newtonfund.ac.uk.

ACKNOWLEDGMENTS

The authors are grateful to the referees for constructive comments and suggestions that have improved this article.

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