Characterization of a New Class of Stochastic Processes Including all Known Extensions of the Class (Σ)

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The class (Σ) is an important family of semimartingales defined by Yor. These processes play a key role in the theory of probability and their applications. For instance, such processes are used to resolve the Skorokhod Imbedding Problem and to construct solutions for homogeneous and inhomogeneous skew Brownian Motion equations. This paper contributes to the study of classes (Σ) and (Σ^r). But, instead of considering as it is customary, the semi-martingales whose finite variational part is continuous, we will consider those whose finite variational part is càdlàg. The two main contributions of this paper are as follows. First, we present a new characterization result for the stochastic processes of class (Σ^r). Second, we provide a framework for unifying the studies of classes (Σ) and (Σ^r). More precisely, we define and study a new larger class that we call class (Σ^g) and for which we give characterization results. In addition, we derive some structural properties inspired of those obtained for classes (Σ) and (Σ^r). Finally, we show that some processes of this new class can take the form of relative martingales. More precisely, we derive a formula allowing to recover some processes of the class (Σ^g) from an honest time and their final value.

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1 Introduction

This study investigates càdlàg (right continuous and admits a left limite) semimartingales of classes (Σ) and (Σ^r). These are stochastic processes X of the following form:

$$X = M + A, \tag{1.1}$$

where M is a càdlàg local martingale with $M_0 = 0$ and A is an adapted predictable process of finite variation with $A_0 = 0$, such that the signed measure induced by A is carried by an optional random set H, where

$$\int_0^t 1_{H^c}(s)dA_s = 0, \forall t \geq 0. \tag{1.2}$$

Such processes are strongly related to many probabilistic studies. Well-known examples of studies where the use of such processes is capitalized include the theory of Azéma–Yor martingales, the study of zeros of continuous martingales [1], the study of Brownian local times, the balayage formulas in the progressive case [2], the construction of solutions for skew Brownian motion equations [3], and the resolution of Skorokhod's reflection equation and embedding problem [4]. These classes are represented in the form of H. More precisely, for the processes of class (Σ), we have

$$H = \{t \geq 0 : X_t = 0\};$$

By contrast, for class (Σ^r), the random set H takes the following form

$$H = \{t \geq 0 : X_{t-} = 0\}.$$

The stochastic processes of class (Σ), whose finite variational part is continuous, have been studied extensively by several authors, including Yor, Najnudel, Nikeghbali, Cheridito, Platen, Ouknine, Bouhadou, Eyi Obiang, Moutsinga, and Trutnau (see [5, 6, 7, 3, 8, 9, 10, 11, 12, 13, 14]). Recently, this notion of class (Σ) has been extended to other areas. For instance, there has been a series of articles in the field of stochastic Calculus for Signed Measures (see [15, 16, 7, 3, 17, 18]) and in the field of Stochastic Calculus for làdlàg semimartingales (see [19]). The authors of all above mentioned references studied the main properties of these processes, presented their applications, and relaxed the original hypotheses. The notion of stochastic processes of class (Σ) has evolved over time, and the present study considers the most general definition presented by Eyi Obiang et al. in [3], which extends the notion of class (Σ) to càdlàg semimartingales, whose finite variational part is considered càdlàg instead of continuous. We consider the following definition:

Definition 1.1. We shall consider that a semimartingale X is of class (Σ) if $X = M + A$, where

1. M is a càdlàg local martingale, with $M_0 = 0$;

2. A is a càdlàg predictable process with finite variations, $A_{0-} = A_0 = 0$;

3. dA is carried by $\{t \geq 0 : X_t = 0\}$.

By contrast, the study of class (Σ^r) is quite recent. In 2018, Akdim et al. [20] first characterized and studied the structural properties of the positive submartingales of the said class. However, it should be noted that the use of the processes of class (Σ^r) has a longer history. For instance, in 1981, Barlow [21] used these processes to show that any positive submartingale is equal to the absolute value of a martingale. More precisely, we consider the following definition:

Definition 1.2. A process X is an element of class (Σ^r) if it $X = M + A$, where

1. M is a càdlàg local martingale vanishing at zero;

2. A is an adapted càdlàg predictable and finite variation process such that $A_{0-} = A_0 = 0$;

3. dA is carried by $\{t \geq 0 : X_{t-} = 0\}$.

Notably, the two above-mentioned classes coincide for the processes X, whose finite variational part A is considered continuous (i.e., class (Σ) under the hypotheses considered by Nikeghbali [12] and Cheridito et al. [6]). However, it is possible to determine processes belonging to at least one of these classes that are not present in another class.

This study contributes toward existing literature by enriching the general framework and developing techniques for dealing with stochastic processes of class (Σ^r) and the càdlàg semimartingales of class (Σ). First, we provide a new way to characterize the processes of class (Σ^r).

Secondly, we present a general framework that unifies the study of classes (Σ) and (Σ^r). More specifically, we propose a new larger class that includes all the processes of the classes (Σ) and (Σ^r). We term this class as (Σ^g) and define it as follows:

Definition 1.3. A process X is an element of the class (Σ^g) if $X = M + V$, where

1. M is a càdlàg local martingale null at zero;

2. V is a càdlàg predictable and finite variation process such that $V_{0-} = V_0 = 0$ and dV is carried by $\{t \geq 0 : X_t X_{t-} = 0\}$.

Hence, we explore and extend known structural properties derived in [12, 6, 13, 20] for classes (Σ) and (Σ^r). In particular, we investigate the positive and negative parts of the processes of class (Σ^g) and show that the multiplication of the processes of class (Σ^g), whose quadratic covariation vanishes, generates an element of class (Σ^g). Further, we obtain a multiplicative decomposition for positive processes of class (Σ^g). Specifically, such processes can be decomposed as

$$X = GW - 1,$$

where W is a positive local martingale with $W_0 = 1$, and G is a non-decreasing process. This result is a generalization of the multiplicative decomposition that Nikeghbali obtained for positive and continuous submartingales [13]. We also present under some assumptions, a formula that enables the recovery of an element X of class (Σ^g) from its final value X_∞ and of an honest time Γ, which is the last time $(X_t : t \geq 0)$ or $(X_{t-} : t \geq 0)$ visited the origin. More precisely, this formula has the following form:

$$X_t = E\left[X_\infty 1_{\{\Gamma \leq t\}} | \mathcal{F}_t\right],$$

where X is the process of class (Σ^g), $X_\infty = \lim_{t \to +\infty} X_t$, and $\Gamma = \sup\{t \geq 0 : X_t X_{t-} = 0\}$. This result generalizes both the representation formulas of the class (Σ) in [6] and those obtained for processes of class (Σ^r) [20]. A particular cas of such formulas appears in works of Madan, Roynette and Yor [22]. Specifically, they proved that

$$(k - M_T)^+ = E\left[(k - M_\infty)^+ 1_{\{\Gamma_k \leq t\}} | \mathcal{F}_t\right],$$

where k is a constant, M is a local martingale with no positive jumps and $\Gamma_k = \sup\{t \geq 0 : M_t \geq k\}$. We extend this formula to all càdlàg local martingales in Section 3. Finally, we derive a series of characterization results for (Σ^g). One of these results generalizes the above mentioned characterization result of the class (Σ^r) and a result characterizing positive sub-martingales of class (Σ) known in the literature as the martingale characterization theorem. This last mentioned result is initially established by Nikeghbali (Theorem 2.1 of [12]) for sub-martingales, whose finite

variational part is continuous and extended to positive special semimartingales of class (Σ), whose finite variational part is càdlàg (see Theorem 2 of [3]).

We organize this paper as follows. In Section 2, we present some useful preliminaries and introduce new characterization of class (Σ^r). Section 3 is devoted to the investigation of the class (Σ^g). Finally, Section 4 summarizes the related approaches and methods.

2 Preliminaries and New Characterization of the Class (Σ^r)

The main purpose of this section is to contribute toward the framework for studying the processes of class (Σ^r). More precisely, we propose a new method for characterizing the positive processes of class (Σ^r). However, we first recall some results and notations that will be useful for understanding this work.

2.1 Notations and preliminaries

In this work, we fix a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}_t,)$ that satisfies the usual conditions. Throughout this work, we use the notation X^c to represent the continuous part of a càdlàg stochastic process X, and $(X_{t-})_{t \geq 0}$ denotes the process defined by $\forall t > 0$, where X_{t-} is the left limit of X in t and $X_{0-} = X_0$.

Now, let us recall the version of class (Σ) studied by Nikeghbali [12] and Cheridito et al. [6]

Definition 2.1. A semi-martingale $X = M + A$ is of class (Σ) if the following holds:

1. M is a càdlàg local martingale null at the origin;
2. A is an adapted continuous, finite variation process, with $A_0 = 0$;
3. $\int_0^t 1_{\{X_s \neq 0\}} dA_s = 0$ for all $t \geq 0$.

Nikeghbali's Theorem 2.1 [12] serves as a method to characterize the non-negative processes that satisfy the assumptions of Definition 2.1. This result is called the characterization martingale theorem. We recall it as follows:

Theorem 2.1. *Let $X = M + A$ be a non-negative semi-martingale. Then, the following are equivalent:*

1. *$X \in (\Sigma)$;*
2. *There exists a non-decreasing continuous process V with $V_0 = 0$ such that, for any locally bounded Borel function f with $F(x) = \int_0^x f(z)dz$, the process*

$$(F(V_t) - f(V_t)X_t; t \geq 0)$$

   is a càdlàg local martingale and $V \equiv A$.

This result was extended by Eyi Obiang et al. [3] for càdlàg non-negative processes satisfying Definition 1.3, in the following way:

Theorem 2.2. *Let $X = M + A$ be a non-negative and càdlàg semi-martingale. Then, the following are equivalent:*

1. *$X \in (\Sigma)$;*

2. *There exists a càdlàg non-decreasing predictable process with finite variations V such that, for any $f \in C^1$ with $F(x) = \int_0^x f(z)dz$, the process*

$$\left(F(V_t^c) - f(V_t^c)X_t + \sum_{s \leq t} [f(V_s^c) - f'(V_s^c)X_s]\Delta V_s; t \geq 0 \right)$$

is a càdlàg local martingale and $V \equiv A$.

Now, recall that the family of processes of class (Σ) considered by Nikeghbali (Definition 2.1) is also included in class (Σ^r). Indeed, it suffices to say that if A is a continuous process, we have

$$\int_0^t 1_{\{X_{s-} \neq 0\}} dA_s = \int_0^t 1_{\{X_s \neq 0\}} dA_s = 0.$$

Hence, in the next subsection, we derive a characterization martingale theorem for the processes of class (Σ^r).

2.2 New characterization result for the class (Σ^r)

Let us start with an extension of Lemma 2.3 of [6].

Lemma 2.3. *Let $X = M + V$ be a process of the class (Σ^r), where V^c denotes the continuous part of V. Let f be a C^1 function and consider the function F defined by $F(x) = \int_0^x f(z)dz$. Hence, the process*

$$\left(F(V_t^c) - f(V_t^c)X_t + \sum_{s \leq t} [f(V_s^c) - f'(V_s^c)X_{s-}]\Delta V_s; t \geq 0 \right)$$

is a càdlàg local martingale.

Proof. Using integration by parts, we get

$$f(V_t^c)X_t = \int_0^t f(V_s^c)dX_s + \int_0^t f'(V_s^c)X_{s-}dV_s^c.$$

Hence, we obtain

$$f(V_t^c)X_t = \int_0^t f(V_s^c)dX_s + \int_0^t f'(V_s^c)X_{s-}dV_s - \sum_{s \leq t} f'(V_t^c)X_{s-}\Delta V_s.$$

But, we have $\int_0^t f'(V_s^c)X_{s-}dV_s = 0$ because dV is carried by $\{t \geq 0 : X_{t-} = 0\}$. Which entails that

$$f(V_t^c)X_t = \int_0^t f(V_s^c)dM_s + \int_0^t f(V_s^c)dV_s^c + \sum_{s \leq t} [f(V_s^c) - f'(V_s^c)X_{s-}]\Delta V_s.$$

We therefore obtain that

$$\left(F(V_t^c) + \sum_{s \leq t} [f(V_s^c) - f'(V_s^c)X_{s-}]\Delta V_s - f(V_t^c)X_t \right)_{t \geq 0}$$

is a local martingale. Which gives the result. $\qquad\square$

Now, we shall derive our martingale characterization theorem for the class (Σ^r).

Theorem 2.4. *Let us consider a positive semi-martingale $X = M + A$. Then, the following are equivalent:*

1. *$X \in (\Sigma^r)$;*
2. *There exists a non-decreasing predictable process V such that, for any $f \in C^1$ and a function F defined by $F(x) = \int_0^x f(s)ds$, the process*

$$\left(F(V_t^c) - f(V_t^c)X_t + \sum_{s \leq t} [f(V_s^c) - f^{'}(V_s^c)X_{s-}]\Delta V_s; t \geq 0 \right)$$

is a càdlàg local martingale and $V \equiv A$.

Proof. We obtain $(1) \Rightarrow (2)$ by applying Lemma 2.3.
$(2) \Rightarrow (1)$ First, let $F(x) = x$. Then, the process

$$W_t = V_t^c + \sum_{s \leq t} \Delta V_s - X_t$$

is a local martingale. Hence, owing to the uniqueness of the special semi-martingale decomposition, we obtain $V = A$. Next, we put $F(x) = x^2$. Thus, process B defined by

$$B_t = (V_t^c)^2 - 2V_t^c X_t + 2\sum_{s \leq t} V_s^c \Delta V_s - 2\sum_{s \leq t} X_{s-}\Delta V_s$$

is a local martingale. However, an application of integration by parts gives the following:

$$B_t = 2 \int_0^t V_s^c dW_s - 2 \int_0^t X_{s-}dV_s.$$

Which implies that

$$\int_0^t X_{s-}dV_s = 0.$$

Consequently, dV is carried by the set $\{t \geq 0 : X_{t-} = 0\}$. $\square$

3 Characterization of the New Class of Stochastic Processes

We propose unifying the study of the stochastic processes of classes (Σ) and (Σ^r). More precisely, we provide a general framework to study a larger class that we term as class (Σ^g).

3.1 First characterization and some properties

As is evident from the above definition, classes (Σ) and (Σ^r) are included in class (Σ^g). Indeed, we can see that $\{X_t = 0\} \subset \{X_t X_{t-} = 0\}$ and $\{X_{t-} = 0\} \subset \{X_t X_{t-} = 0\}$. However, there exist processes of class (Σ^g) that do not belong to classes (Σ) and (Σ^r). For instance, if M is a càdlàg local martingale, M^+ and M^- are elements of the class (Σ^g) (see Lemma 3.8). However, $M^+ \in (\Sigma^r)$ only if, M has no negative jump and $M^+ \in (\Sigma)$ only if, M has no positive jump. Next, we present the first characterization result for the class (Σ^g).

Theorem 3.1. *Let us consider a càdlàg semi-martingale $X = M + A$. Then, the following are equivalent:*

1. *$X \in (\Sigma^g)$;*
2. *there exist two predictable processes C and V such that $A = C + V$ and*

$$\int_0^t 1_{\{X_s \neq 0\}} dC_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = 0.$$

Proof. $(1) \Rightarrow (2)$ We can see that, for all $t \geq 0$,

$$A_t = \int_0^t 1_{\{X_s = 0\}} dA_s + \int_0^t 1_{\{X_s X_{s-} \neq 0\}} dA_s + \int_0^t 1_{\{X_s \neq 0, X_{s-} = 0\}} dA_s.$$

However, $\int_0^t 1_{\{X_s X_{s-} \neq 0\}} dA_s = 0$ as dA_s is carried by $\{X_s X_{s-} = 0\}$. Hence, it entails the following:

$$A_t = C_t + V_t,$$

where $C_t = \int_0^t 1_{\{X_s = 0\}} dA_s$ and $V_t = \int_0^t 1_{\{X_s \neq 0, X_{s-} = 0\}} dA_s$. Thus, we obtain $\forall t \geq 0$,

$$\int_0^t 1_{\{X_s \neq 0\}} dC_s = \int_0^t 1_{\{X_s \neq 0\}} 1_{\{X_s = 0\}} dA_s = 0 \text{ and } \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = \int_0^t 1_{\{X_{s-} \neq 0\}} 1_{\{X_s \neq 0, X_{s-} = 0\}} dA_s = 0.$$

$(2) \Rightarrow (1)$

Now, assume that $A = C + V$ with $\int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = \int_0^t 1_{\{X_s \neq 0\}} dC_s = 0$. One has $\forall t \geq 0$,

$$\int_0^t 1_{\{X_s X_{s-} \neq 0\}} dA_s = \int_0^t 1_{\{X_s X_{s-} \neq 0\}} dC_s + \int_0^t 1_{\{X_s X_{s-} \neq 0\}} dV_s.$$

However,

$$\int_0^t 1_{\{X_s X_{s-} \neq 0\}} dC_s = \int_0^t 1_{\{X_{s-} \neq 0\}} 1_{\{X_s \neq 0\}} dC_s = 0, \text{ since } 1_{\{X_s \neq 0\}} dC_s \equiv 0$$

and

$$\int_0^t 1_{\{X_s X_{s-} \neq 0\}} dV_s = \int_0^t 1_{\{X_s \neq 0\}} 1_{\{X_{s-} \neq 0\}} dV_s = 0, \text{ since } 1_{\{X_{s-} \neq 0\}} dV_s \equiv 0.$$

Which gives the result. $\qquad\square$

Now, we derive two corollaries that provide a new approach to characterize the classes (Σ) and (Σ^r).

Corollary 3.2. *Let us consider a càdlàg semi-martingale $X = M + A$. Then, the following are equivalent:*

1. *$X \in (\Sigma)$;*
2. *there exist a continuous finite variation process V and a càdlàg predictable process C such that $A = C + V$ and $\int_0^t 1_{\{X_s \neq 0\}} dC_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = 0$.*

Proof. $(1) \Rightarrow (2)$ Assume that X is an element of the class (Σ). Hence, it follows from Definition 1.3 that there exists a local martingale M and a càdlàg, predictable process A such that $\forall t \geq 0$, dA_t is carried by $\{X_t = 0\}$ and $X = M + A$. It is evident that (2) yields by taking $C = A$ and $V \equiv 0$.

$(2) \Rightarrow (1)$ Now, assume that Assertion (2) is true. We have $\forall t \geq 0$,

$$\int_0^t 1_{\{X_s \neq 0\}} dA_s = \int_0^t 1_{\{X_s \neq 0\}} dC_s + \int_0^t 1_{\{X_s \neq 0\}} dV_s.$$

However, $\int_0^t 1_{\{X_s \neq 0\}} dC_s = 0$ as dC_s is carried by $\{X_s \neq 0\}$. Hence, $\int_0^t 1_{\{X_s \neq 0\}} dA_s = \int_0^t 1_{\{X_s \neq 0\}} dV_s$. Furthermore, $\int_0^t 1_{\{X_s \neq 0\}} dV_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s$ because V is continuous. Therefore,

$$\int_0^t 1_{\{X_s \neq 0\}} dA_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = 0.$$

Consequently, X is an element of the class (Σ). This completes the proof. $\qquad\square$

Corollary 3.3. *Let $X = M + A$ be a càdlàg stochastic process. Then, the following are equivalent:*

1. $X \in (\Sigma^r)$;

2. *there exist a continuous finite variation process C and a càdlàg predictable process V such that $A = C + V$ and $\int_0^t 1_{\{X_s \neq 0\}} dC_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = 0$.*

Proof. $(1) \Rightarrow (2)$ Assume that X is an element of class (Σ^r). Hence, there exist a local martingale M and a càdlàg predictable process A such that $\forall t \geq 0$, dA_t is carried by $\{X_{t-} = 0\}$ and $X = M + A$. It is clear that (2) yields by taking $V = A$ and $C \equiv 0$.
$(2) \Rightarrow (1)$ Now, assume that Assertion (2) is true. We have $\forall t \geq 0$,

$$\int_0^t 1_{\{X_{s-} \neq 0\}} dA_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dC_s + \int_0^t 1_{\{X_{s-} \neq 0\}} dV_s.$$

However, $\int_0^t 1_{\{X_{s-} \neq 0\}} dV_s = 0$ as dV_s is carried by $\{X_{s-} \neq 0\}$. Hence,

$$\int_0^t 1_{\{X_{s-} \neq 0\}} dA_s = \int_0^t 1_{\{X_{s-} \neq 0\}} dC_s.$$

Furthermore, $\int_0^t 1_{\{X_{s-} \neq 0\}} dC_s = \int_0^t 1_{\{X_s \neq 0\}} dC_s$ because C is continuous. Therefore,

$$\int_0^t 1_{\{X_{s-} \neq 0\}} dA_s = \int_0^t 1_{\{X_s \neq 0\}} dC_s = 0.$$

Consequently, X is in the class (Σ^r). This completes the proof. $\qquad\square$

Now, we shall derive general properties of the stochastic processes of the class (Σ^g). Hence, we begin by deriving the properties using the balayage formulas:

Lemma 3.4. *Let us consider a process X the of class (Σ^g), and let $\gamma_t = \sup\{s \leq t : X_s = 0\}$. Then, for any bounded predictable process k, $k_{\gamma.} X$ is also an element of class (Σ^g).*

Proof. Through balayage's formula for the càdlàg case, we obtain the following:

$$k_{\gamma_t} X_t = k_{\gamma_0} X_0 + \int_0^t k_{\gamma_s} dX_s = \int_0^t k_{\gamma_s} dM_s + \int_0^t k_{\gamma_s} dC_s + \int_0^t k_{\gamma_s} dV_s.$$

It is clear that $\int_0^{\cdot} k_{\gamma_s} dM_s$ is a local martingale; furthermore, $k dC_t$ is carried by $\{t \geq 0 : k_{\gamma_t} X_t = 0\}$ and $k dV_t$ is carried by $\{t \geq 0 : k_{\gamma_{t-}} X_{t-} = 0\}$. This completes the proof. $\qquad\square$

Corollary 3.5. *Let us consider $X = M + C + V = M + A$, a process of class (Σ^g), and let f be a bounded Borel function. Then, the process $(f(C_t) X_t : t \geq 0)$ is an element of the class (Σ^g), and its finite variation part is defined by $\forall t \geq 0$, $A'_t = \int_0^t f(C_s) d(C_s + V_s)$.*

Proof. According to Lemma 3.4, $(f(C_{\gamma_t})X_t : t \geq 0)$ is an element of the class (Σ^g). Furthermore, we have $\forall t \geq 0$, $f(C_{\gamma_t})X_t = \int_0^t f(C_{\gamma_s})dM_s + \int_0^t f(C_{\gamma_s})dA_s$. As dC_t is carried by $\{X_t = 0\}$, we have $\forall t \geq 0$, $C_{\gamma_t} = C_t$. Consequently, $\forall t \geq 0$, $f(C_t)X_t = \int_0^t f(C_s)dM_s + \int_0^t f(C_s)dA_s$. This completes the proof. □

Corollary 3.6. *Let $X = M+C+V$ be a positive process of the class (Σ^g). Then, there exist a càdlàg non-decreasing predictable process Γ satisfying $Supp(d\Gamma_t) \subset \{X_t = 0\}$ and a positive submartingale $W = m + l$ with $W_0 = 1$; the measure dl_t is carried by $\{X_{t-} = 0\}$ such that $\forall t \geq 0$, $X_t = \Gamma_t W_t - 1$.*

Proof. It follows from Corollary 3.5 that $f(C_t)X_t - \int_0^t f(C_s)dC_s = \int_0^t f(C_s)dM_s + \int_0^t f(C_s)dV_s$, since the function f, defined by $f(x) = e^{-x}$, is a bounded Borel function on $[0, +\infty[$. Hence, we obtain that, $\forall t \geq 0$,

$$e^{-C_t}(X_t + 1) - 1 = \int_0^t e^{-C_s}dM_s + \int_0^t e^{-C_s}dV_s.$$

Therefore, considering $W_t = 1 + \int_0^t e^{-C_s}dM_s + \int_0^t e^{-C_s}dV_s$, we get

$$e^{-C_t}(X_t + 1) = W_t. \tag{3.1}$$

Consequently, $X_t = \Gamma_t W_t - 1$, where $\Gamma_t = e^{C_t}$. It is evident from (3.1) that W is a positive submartingale with $W_0 = 1$, and its non-decreasing part $l = \int_0^t e^{-C_s}dV_s$ is such that $Supp(dl_t) \subset \{X_{t-} = 0\}$. □

Now, we investigate properties satisfied by the negative and positive parts of the stochastic processes of the class (Σ^g).

Lemma 3.7. *Let $X = M + C + V$ be in the class (Σ^g). The following hold:*

1. *If C is a non-decreasing process, then X^+ is a local submartingale.*
2. *If C is a decreasing process, then X^- is a local submartingale.*
3. *If C has no negative jump and $\int_0^t 1_{\{X_s \neq 0\}}dC_s^c = 0$, then X^+ is a local submartingale.*
4. *If C has no positive jump and $\int_0^t 1_{\{X_s \neq 0\}}dC_s^c = 0$, then X^- is a local submartingale.*

Proof. Through Tanaka's formula, we get

$$X_t^+ = \int_0^t 1_{\{X_{s-}>0\}}dX_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}}X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}}X_s^- + \frac{1}{2}L_t^0.$$

On another hand, $\int_0^t 1_{\{X_{s-}>0\}}dX_s = \int_0^t 1_{\{X_{s-}>0\}}dM_s + \int_0^t 1_{\{X_{s-}>0\}}dC_s + \int_0^t 1_{\{X_{s-}>0\}}dV_s$. Hence,

$$\int_0^t 1_{\{X_{s-}>0\}}dX_s = \int_0^t 1_{\{X_{s-}>0\}}dM_s + \int_0^t 1_{\{X_{s-}>0\}}dC_s$$

as $\int_0^t 1_{\{X_{s-}>0\}}dV_s = 0$; this is because dV_t is carried by $\{X_{t-} = 0\}$. Then,

$$X_t^+ = \int_0^t 1_{\{X_{s-}>0\}}dM_s + \int_0^t 1_{\{X_{s-}>0\}}dC_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}}X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}}X_s^- + \frac{1}{2}L_t^0. \tag{3.2}$$

Thus, we have the following:

1. We first remark that

$$\left(\int_0^t 1_{\{X_{s-}>0\}} dC_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}} X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} X_s^- + \frac{1}{2} L_t^0; t \geq 0 \right)$$

is an increasing process that is null at zero, since C is a non-decreasing process.
Furthermore, M and $\int_0^\cdot 1_{\{X_{s-}>0\}} dM_s$ are local martingales. Then, X^+ is a local submartingale.

2. Now, for any process of the class (Σ^g), $-X$ is again an element of the class (Σ^g).
Therefore, $X^- = (-X)^+$ is a local submartingale when the process C decreases.

3. We obtain the following from identity (3.2):

$$X_t^+ = \int_0^t 1_{\{X_{s-}>0\}} dM_s + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} \Delta C_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}} X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} X_s^- + \frac{1}{2} L_t^0,$$

as

$$\int_0^t 1_{\{X_{s-}>0\}} dC_s = \int_0^t 1_{\{X_{s-}>0\}} dC_s^c + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} \Delta C_s \text{ and } \int_0^t 1_{\{X_{s-}>0\}} dC_s^c = \int_0^t 1_{\{X_s>0\}} dC_s^c = 0.$$

Hence,

$$\left(\sum_{0<s\leq t} 1_{\{X_{s-}>0\}} \Delta C_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}} X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} X_s^- + \frac{1}{2} L_t^0; t \geq 0 \right)$$

is an increasing process because C has no negative jump. Consequently, X^+ is a local submartingale.

$\square$

Remark 3.1. A direct consequence is that any non-negative stochastic process of class (Σ^g) satisfying the assumptions of Lemma 3.7 is a submartingale.

Lemma 3.8. *Let X be a process of class (Σ^g). Hence, X^+ and X^- are stochastic processes of class (Σ^g).*

Proof. Based on Tanaka's formula, we have

$$X_t^+ = \int_0^t 1_{\{X_{s-}>0\}} dX_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}} X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} X_s^- + \frac{1}{2} L_t^0.$$

However,

$$\int_0^t 1_{\{X_{s-}>0\}} dX_s = \int_0^t 1_{\{X_{s-}>0\}} dM_s + \int_0^t 1_{\{X_{s-}>0\}} dC_s + \int_0^t 1_{\{X_{s-}>0\}} dV_s = \int_0^t 1_{\{X_{s-}>0\}} dM_s + \int_0^t 1_{\{X_{s-}>0\}} dC_s,$$

as dV_t is carried by $\{X_{t-}=0\}$. Hence,

$$X_t^+ = \int_0^t 1_{\{X_{s-}>0\}} dM_s + \int_0^t 1_{\{X_{s-}>0\}} dC_s + \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}} X_s^+ + \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} X_s^- + \frac{1}{2} L_t^0. \quad (3.3)$$

Now, let's put $Y_t = \sum_{0<s\leq t} 1_{\{X_{s-}\leq 0\}} X_s^+$ and $Z_t = \sum_{0<s\leq t} 1_{\{X_{s-}>0\}} X_s^-$. Hence, there exists a sequence of stopping times $(T_n; n \in)$ increasing to ∞, such that

$$E[(X_{T_n})^+] = E[(M_{T_n} + C_{T_n} + V_{T_n})^+] < \infty \text{ and } E\left[\int_0^{T_n} 1_{\{X_{s-}>0\}} dM_s \right] = 0, \, n \in,$$

since M and $\int_0^{\cdot} 1_{\{X_{s-}>0\}} dM_s$ are local martingales and $C+V$ is càdlàg. we get from Equation (3.3) that $E[Y_{T_n}] \leq E\left[(X_{T_n})^+ - \int_0^t 1_{\{X_{s-}>0\}} dC_s\right] < \infty$ and

$$E[Z_{T_n}] \leq E\left[(X_{T_n})^+ - \int_0^t 1_{\{X_{s-}>0\}} dC_s\right] < \infty$$

for all $n \in$. Thus, based on Theorem VI.80 of [23], there exist right continuous increasing predictable processes V^Y and V^Z such that $Y - V^Y$ and $Z - V^Z$ are local martingales vanishing at zero. Moreover, there exists a sequence of stopping times $(R_n; n \in)$ increasing to ∞, such that

$$E\left[\int_0^{t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} dV_s^Y\right] = E\left[\int_0^{t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} d(V_s^Y - Y_s) + \int_0^{t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} dY_s\right].$$

As $\int_0^{\cdot \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} d(V_s^Y - Y_s)$ is a local martingale, it entails that

$$E\left[\int_0^{t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} dV_s^Y\right] = E\left[\int_0^{t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} dY_s\right].$$

Therefore,

$$E\left[\int_0^{t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} dV_s^Y\right] = E\left[\sum_{0 < s \leq t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} 1_{\{X_{s-} \leq 0\}} X_s^+\right] = E\left[\sum_{0 < s \leq t \wedge R_n} 1_{\{X_{s-}^+ \neq 0\}} 1_{\{X_{s-}^+ = 0\}} X_s^+\right] = 0.$$

In other words, $\int_0^t 1_{\{X_{s-}^+ \neq 0\}} dV_s^Y = 0$. Then, dV_t^Y is carried by $\{X_{t-}^+ = 0\}$. However, we have

$$E\left[\int_0^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dV_s^Z\right] = E\left[\int_0^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} d(V_s^Z - Z_s) + \int_0^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dB_s\right]$$

$$= E\left[\int_0^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} a\,dz_s\right].$$

Hence,

$$E\left[\int_0^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dV_s^Z\right] = E\left[\sum_{0 < s \leq t \wedge R_n} 1_{\{X_s^+ \neq 0\}} 1_{\{X_{s-} > 0\}} X_s^-\right] = E\left[\sum_{0 < s \leq t \wedge R_n} 1_{\{X_s > 0\}} 1_{\{X_{s-} > 0\}} X_s^-\right].$$

This entails that

$$E\left[\int_0^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dV_s^Z\right] = 0,$$

since $1_{\{X_s > 0\}} X_s^- = 0$. This shows that $\int_0^t 1_{\{X_s^+ \neq 0\}} dV_s^Z = 0$. Therefore, dV_t^Z is carried by $\{t \geq 0; X_t^+ = 0\}$. Consequently, we determine that

$$X_t^+ = \left(\int_0^t 1_{\{X_{s-}>0\}} dM_s + (Y_t - V_t^Y) + (Z_t - V_t^Z)\right) + V_t^Y + \left(V_t^Z + \int_0^t 1_{\{X_{s-}>0\}} dC_s + \frac{1}{2} L_t^0\right)$$

is a stochastic process of the class (Σ^g). This is also true for X^- as $(-X)$ is also from class (Σ^g). $\square$

It is well known that M^+ and M^- are stochastic processes of class (Σ) when M is a continuous local martingale. The next corollary of Lemma 3.8 shows that M^+ and M^- are elements of class (Σ^g) when M is a càdlàg local martingale.

Corollary 3.9. *Let M be a càdlàg local martingale vanishing at zero. Then, the processes M^+ and M^- are elements of class (Σ^g).*

Now, we show that the product of the processes of class (Σ^g) with vanishing quadratic covariations is again of class (Σ^g).

Lemma 3.10. *Let $(X_t^1)_{t \geq 0}, \cdots, (X_t^n)_{t \geq 0}$ be elements of class (Σ^g) satisfying the following $[X^i, X^j] = 0$ for $i \neq j$. Then, $(\Pi_{i=1}^n X_t^i)_{t \geq 0}$ is also of class (Σ^g).*

Proof. As $[X^1, X^2] = 0$, it follows by integration by parts that

$$X_t^1 X_t^2 = \int_0^t X_{s-}^1 \, dX_s^2 + \int_0^t X_{s-}^2 \, dX_s^1.$$

In other words,

$$X_t^1 X_t^2 = \left[\int_0^t X_{s-}^1 \, dM_s^2 + \int_0^t X_{s-}^2 \, dM_s^1 \right] + \left[\int_0^t X_{s-}^1 \, dC_s^2 + \int_0^t X_{s-}^2 \, dC_s^1 \right] + \left[\int_0^t X_{s-}^1 \, dV_s^2 + \int_0^t X_{s-}^2 \, dV_s^1 \right].$$

It can be observed that $M_t = \int_0^t X_{s-}^1 \, dM_s^2 + \int_0^t X_{s-}^2 \, dM_s^1$ is a càdlàg local martingale. Furthermore, the process $C_t = \int_0^t X_{s-}^1 \, dC_s^2 + \int_0^t X_{s-}^2 \, dC_s^1$ is a finite variation process, such that

$$dC_t = X_{t-}^1 \, dC_t^2 + X_{t-}^2 \, dC_t^1$$

is carried by $\{t \geq 0 : X_t^1 X_t^2 = 0\}$. By contrast, $V_t = \int_0^t X_{s-}^1 \, dV_s^2 + \int_0^t X_{s-}^2 \, dV_s^1$ is a finite variation process, such that

$$dV_t = X_{t-}^1 \, dV_t^2 + X_{t-}^2 \, dV_t^1$$

is carried by $\{t \geq 0 : X_{t-}^1 X_{t-}^2 = 0\}$. Therefore, $X^1 X^2$ is of class (Σ^g). If $n \geq 3$, and $[X^1 X^2, X^3] = 0$. Thus, we obtain the result by induction. $\square$

Definition 3.1. A stochastic process X is said of class if $\{X_\tau : \tau < \infty \text{ is a stopping time}\}$ is uniformly integrable.

Theorem 3.11. *Let $X = M + C + V$ be a process of class (Σ^g) and an element of class . Then, there exists a random variable X_∞ such that*

$$\lim_{t \to +\infty} X_t = X_\infty$$

, and for every stopping time $T < \infty$, we have

$$X_T = E \left[X_\infty 1_{\{g < T\}} | \mathcal{F}_T \right], \tag{3.4}$$

where $g = \sup \{t \geq 0 : X_t X_{t-} = 0\}$.

Proof. Let us substitute $\gamma_t = \inf \{s > t \geq 0 : X_s X_{s-} = 0\}$. It is evident that γ_t is the stopping time. Furthermore,

$$X_\infty 1_{\{g < T\}} = X_{\gamma_T} = M_{\gamma_T} + C_{\gamma_T} + V_{\gamma_T}.$$

However, $C_{\gamma_T} = C_T$ and $V_{\gamma_T} = V_T$ as dC and dV are carried by $\{t \geq 0 : X_t = 0\}$ and $\{t \geq 0 : X_{t-} = 0\}$, respectively; further, $g = \sup \{t \geq 0 : X_t = 0\} \vee \sup \{t \geq 0 : X_{t-} = 0\}$. This entails that

$$X_\infty 1_{\{g < T\}} = X_{\gamma_T} = M_{\gamma_T} + C_T + V_T.$$

Hence,

$$E \left[X_\infty 1_{\{g < T\}} | \mathcal{F}_T \right] = E [M_{\gamma_T} | \mathcal{F}_T] + C_T + V_T.$$

Therefore,

$$X_T = E \left[X_\infty 1_{\{g < T\}} | \mathcal{F}_T \right]$$

as M is a uniformly integrable martingale. $\square$

Corollary 3.12. *Let M be a non-negative càdlàg uniformly integrable martingale such that $M_0 > 0$ and $\lim_{t \to +\infty} M_t = 0$ and $k > 0$ be a constant. Then,*

$$P\left(g_k \geq t | \mathcal{F}_t\right) = 1 \wedge \left(\frac{M_t}{k}\right),$$

where $g_k = \sup\left\{t \geq 0 : M_t \geq k \text{ or } M_{t-} \geq k\right\}$

Proof. It follows from Theorem 2.4 that

$$\left(k - M_t\right)^+ = E\left[k 1_{\{g_k < t\}} | \mathcal{F}_t\right] = k E\left[1_{\{g_k < t\}} | \mathcal{F}_t\right].$$

Hence,

$$\left(k - M_t\right)^+ = k P\left(g_k < t | \mathcal{F}_t\right).$$

Consequently,

$$P\left(g_k < t | \mathcal{F}_t\right) = \left(1 - \frac{M_t}{k}\right)^+.$$

Therefore,

$$P\left(g_k \geq t | \mathcal{F}_t\right) = 1 - \left(1 - \frac{M_t}{k}\right)^+ = 1 \wedge \left(\frac{M_t}{k}\right).$$

This completes the proof. $\qquad\square$

3.2 Extension of characterization martingale

Lemma 3.13. *Let $X = M + A$ be a process of class (Σ^g), where $A = C + V$ and A^c denote the continuous part of A. Then, for every $\mathcal{C}^1$ function f and $F(x) = \int_0^x f(z)dz$, the process*

$$\left(F(A_t^c) - f(A_t^c)X_t + \sum_{s \leq t}[f(A_s^c) - f'(A_s^c)X_s]\Delta C_s + \sum_{s \leq t}[f(A_s^c) - f'(A_s^c)X_{s-}]\Delta V_s; t \geq 0\right)$$

is a local martingale.

Proof. Integration by parts yields

$$f(A_t^c)X_t = \int_0^t f(A_s^c)dX_s + \int_0^t f'(A_s^c)X_{s-}dA_s^c.$$

In other words,

$$f(A_t^c)X_t = \int_0^t f(A_s^c)dX_s + \int_0^t f'(A_s^c)X_{s-}dC_s^c + \int_0^t f'(A_s^c)X_{s-}dV_s^c$$

since $A^c = C^c + V^c$. Furthermore,

$$\int_0^t f'(A_s^c)X_{s-}dC_s^c = \int_0^t f'(A_s^c)X_s dC_s^c$$

because C^c is continuous. Therefore, we obtain

$$f(A_t^c)X_t = \int_0^t f(A_s^c)dX_s + \int_0^t f'(A_s^c)X_s dC_s^c + \int_0^t f'(A_s^c)X_{s-}dV_s^c.$$

This entails the following:

$$f(A_t^c)X_t = \int_0^t f(A_s^c)dX_s + \left[\int_0^t f^{'}(A_s^c)X_s dC_s - \sum_{s \leq t} f^{'}(A_s^c)X_s \Delta C_s\right] + \left[\int_0^t f^{'}(A_s^c)X_{s-}dV_s - \sum_{s \leq t} f^{'}(A_s^c)X_{s-}\Delta V_s\right].$$

Thus, it follows that

$$f(A_t^c)X_t = \int_0^t f(A_s^c)dX_s - \sum_{s \leq t} f^{'}(A_s^c)X_s \Delta C_s - \sum_{s \leq t} f^{'}(A_s^c)X_{s-}\Delta V_s$$

as dC and dV are carried by $\{t \geq 0 : X_t = 0\}$ and $\{t \geq 0 : X_{t-} = 0\}$, respectively. This entails that

$$f(A_t^c)X_t = \int_0^t f(A_s^c)dM_s + \int_0^t f(A_s^c)dA_s^c + \sum_{s \leq t}\left[f(A_s^c) - f^{'}(A_s^c)X_s\right]\Delta C_s + \sum_{s \leq t}\left[f(A_s^c) - f^{'}(A_s^c)X_{s-}\right]\Delta V_s.$$

Consequently,

$$F(A_t^c) - f(A_t^c)X_t + \sum_{s \leq t}\left[f(A_s^c) - f^{'}(A_s^c)X_s\right]\Delta C_s + \sum_{s \leq t}\left[f(A_s^c) - f^{'}(A_s^c)X_{s-}\right]\Delta V_s = -\int_0^t f(A_s^c)dM_s.$$

In other words,

$$\left(F(A_t^c) - f(A_t^c)X_t + \sum_{s \leq t}[f(A_s^c) - f^{'}(A_s^c)X_s]\Delta C_s + \sum_{s \leq t}[f(A_s^c) - f^{'}(A_s^c)X_{s-}]\Delta V_s; t \geq 0\right)$$

is a local martingale. $\qquad\square$

Theorem 3.14. *Let $X = M + A$ be a positive semi-martingale. Then, the following are equivalent:*

1. *$X \in (\Sigma^g)$;*

2. *There exists two càdlàg and non-decreasing predictable processes V and C such that, for $W = C + V$ and for any $f \in C^1$ and $F(x) =\in_0^x f(s)ds$, the process*

$$\left(F(W_t^c) - f(W_t^c)X_t + \sum_{s \leq t}[f(W_s^c) - f^{'}(W_s^c)X_s]\Delta C_s + \sum_{s \leq t}[f(W_s^c) - f^{'}(W_s^c)X_{s-}]\Delta V_s; t \geq 0\right)$$

is a càdlàg local martingale and $W \equiv A$.

Proof. $(1) \Rightarrow (2)$ By putting $W = A$, we obtain from Lemma 3.13 that

$$\left(F(A_t^c) - f(A_t^c)X_t + \sum_{s \leq t}[f(A_s^c) - f^{'}(A_s^c)X_s]\Delta C_s + \sum_{s \leq t}[(A_s^c) - f^{'}(A_s^c)X_{s-}]\Delta V_s; t \geq 0\right)$$

is a càdlàg local martingale.
$(2) \Rightarrow (1)$ First, let $F(x) = x$. Thus, the process

$$W_t^{'} = W_t^c + \sum_{s \leq t}\Delta C_s + \sum_{s \leq t}\Delta V_s - X_t = W_t - X_t$$

is a local martingale. Thus, owing to the uniqueness of the special semimartingale's decomposition, we obtain $A = W$. Next, we put $F(x) = x^2$. Then, the process B defined by

$$B_t = (W_t^c)^2 - 2W_t^c X_t + 2\sum_{s \leq t}W_s^c \Delta C_s + 2\sum_{s \leq t}W_s^c \Delta V_s - 2\sum_{s \leq t}X_s \Delta C_s - 2\sum_{s \leq t}X_{s-}\Delta V_s$$

is a local martingale. In addition, through integration by part, it follows that

$$B_t = 2\int_0^t W_s^c dW_s^c - 2\int_0^t W_s^c dX_s - 2\int_0^t X_s dW_s^c + 2\sum_{s\leq t} W_s^c \Delta C_s + 2\sum_{s\leq t} W_s^c \Delta V_s - 2\sum_{s\leq t} X_s \Delta C_s - 2\sum_{s\leq t} X_{s-} \Delta V_s$$

$$= 2\int_0^t W_s^c dW_s^c - 2\int_0^t W_s^c dX_s - 2\int_0^t X_s dC_s^c - 2\int_0^t X_s dV_s^c + 2\sum_{s\leq t} W_s^c \Delta C_s + 2\sum_{s\leq t} W_s^c \Delta V_s - 2\sum_{s\leq t} X_s \Delta C_s - 2\sum_{s\leq t} X_{s-} \Delta V_s$$

$$= 2\int_0^t W_s^c dW_s^c - 2\int_0^t W_s^c dX_s - 2\int_0^t X_s dC_s^c - 2\int_0^t X_{s-} dV_s^c + 2\sum_{s\leq t} W_s^c \Delta C_s + 2\sum_{s\leq t} W_s^c \Delta V_s - 2\sum_{s\leq t} X_s \Delta C_s - 2\sum_{s\leq t} X_{s-} \Delta V_s$$

$$= 2\int_0^t W_s^c d\left(W_s^c + \sum_{u\leq s}\Delta C_u + \sum_{u\leq s}\Delta V_u - X_s\right) - 2\int_0^t X_s d\left(C_s^c + \sum_{u\leq s}\Delta C_u\right) - 2\int_0^t X_{s-} d\left(V_s^c + \sum_{u\leq s}\Delta V_u\right)$$

$$= 2\int_0^t W_s^c dW_s' - 2\int_0^t X_s dC_s - 2\int_0^t X_{s-} dV_s.$$

Consequently, we have $\int_0^t X_s dC_s + \int_0^t X_{s-} dV_s = 0$. Hence, we determine that

$$\int_0^t X_s dC_s = \int_0^t X_{s-} dV_s = 0,$$

as $\int_0^t X_s dC_s$ and $\int_0^t X_{s-} dV_s$ are non-negative. Which means that dA is carried by the set $\{t \geq 0 : X_t X_{t-} = 0\}$.

$\square$

4 Conclusion

This study contributes to the development of a general framework and techniques for dealing with stochastic processes of classes which extend the notion of class (Σ) to cádlág semimartingales, whose the finite variational part is also cádlág instead of continuous. First, we have proposed a new method to characterize stochastic processes of the class (Σ^r). Second, we have unified the study of the class (Σ^r) and semimartingales of class (Σ), whose the finite variational part is cádlág by studying a new larger family that we term class (Σ^g). Hence, we have explored some general and structural properties. We have also presented a result that enables the recovery of any process of class (Σ^g) from its final value X_∞ and of an honest time g, which is the last time $(X_t : t \geq 0)$ or $(X_{t-} : t \geq 0)$ visited the origin. More precisely, this formula has the following form:

$$X_t = E\left[X_\infty 1_{\{g\leq t\}}|\mathcal{F}_t\right],$$

where X is the process of class (Σ^g), $X_\infty = \lim_{t\to+\infty} X_t$, and $g = \sup\{t \geq 0 : X_t X_{t-} = 0\}$. Finally, we derived some characterization results which generalize those already existing for the classes (Σ) and (Σ^r).

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Competing Interests

Authors have declared that no competing interests exist.

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