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Research Article

Optimal $L^p - L^q$-Type Decay Rates of Solutions to the Three-Dimensional Nonisentropic Compressible Euler Equations with Relaxation

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In this paper, we consider the three-dimensional Cauchy problem of the nonisentropic compressible Euler equations with relaxation. Following the method of Wu et al. (2021, Adv. Math. Phys. Art. ID 5512285, pp. 1–13), we show the existence and uniqueness of the global small $H^k (k \geqslant 3)$ solution only under the condition of smallness of the H^3 norm of the initial data. Moreover, we use a pure energy method with a time-weighted argument to prove the optimal $L^p - L^q$ $(1 \leqslant p \leqslant 2, 2 \leqslant q \leqslant \infty)$-type decay rates of the solution and its higher-order derivatives.

1. Introduction

In this paper, we shall study the nonisentropic compressible Euler equations with relaxation (cf. [1]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P = -\dfrac{1}{\tau}\rho u, \\ (\rho \mathscr{E})_t + \operatorname{div}(\rho u \mathscr{E} + uP) = -\dfrac{1}{\tau}\rho u^2 + \dfrac{\rho}{\tau}(g(\rho) - \theta). \end{cases} \tag{1}$$

Here, $(x, t) \in \mathbb{R}^3 \times [0,\infty)$, and the unknown variables $\rho = \rho(x, t)$, $u = u(x, t)$, $\theta = \theta(x, t)$, and $P = P(x, t)$ denote the density, the velocity, the absolute temperature, and the pressure, respectively. The total energy per unit mass $\mathscr{E} = 1/2 |u|^2 + e$, and e is the internal energy per unit mass. The constant $\tau > 0$ is the relaxation parameter. The function $g(\rho)$ is smooth with respect to ρ. The system (1) can be used to model a compressible gas flow through a porous medium [2–4]. Assuming that the gas is perfect and polytropic, then

$$\begin{aligned} P &= \rho\theta, \\ e &= \frac{1}{\gamma - 1}\theta, \end{aligned} \tag{2}$$

where $\gamma > 1$ is the adiabatic exponent. Using the constitutive relations (2), the system (1) is reduced to

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ u_t + u \cdot \nabla u + \theta \dfrac{\nabla \rho}{\rho} + \nabla \theta = -\dfrac{1}{\tau} u, \\ \dfrac{1}{\gamma - 1}\theta_t + \dfrac{1}{\gamma - 1} u \cdot \nabla \theta + \theta \operatorname{div} u = \dfrac{1}{\tau}(g(\rho) - \theta). \end{cases} \tag{3}$$

We supplement (3) with the initial condition

$$(\rho, u, \theta)(x, t)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \longrightarrow (1, 0, 1), \quad |x| \longrightarrow \infty. \tag{4}$$

Now, we review the known research results for the compressible Euler equations with relaxation. When considering the isentropic or isothermal case, the system (1) becomes

$$\begin{cases} \rho_t + \operatorname{div}\ (\rho u) = 0, \\ (\rho u)_t + \operatorname{div}\ (\rho u \otimes u) + \nabla P = -\frac{1}{\tau}\rho u. \end{cases} \quad (5)$$

There are a lot of research works on the system (5). For the one-dimensional Cauchy problem, one can refer to [5, 6] for the existence of the global BV solutions, to [7–12] for the global L^∞ entropy-weak solutions with vacuum, and to [13, 14] for small smooth solutions. For the one-dimensional initial-boundary value problem, one can refer to [15, 16] for the existence of the global L^∞ entropy-weak solutions and to [17–19] for small smooth solutions. For the asymptotics of solutions, we refer to [9–12] for L^∞ entropy-weak solutions and to [14, 20–22] for small smooth solutions. In addition, there are some results on the one-dimensional nonisentropic compressible Euler equations with relaxation (cf. [1–4, 23, 24]). The global existence and large-time behavior of solutions to the multidimensional isentropic compressible Euler equations with relaxation were studied by many researchers (cf. [16, 25–36] and the references cited therein).

To the best of our knowledge, there are few results on the three-dimensional nonisentropic compressible Euler equations with relaxation (1). In this paper, following the similar discussions in [37], we shall use a delicate energy method to obtain a refined global existence and uniqueness result, in which we only require the initial H^3 norm to be small. Moreover, we will prove the optimal $L^p - L^q (1 \leqslant p \leqslant 2, 2 \leqslant q \leqslant \infty)$ -type decay rates of solutions as well as its higher-order derivatives by employing the negative Sobolev or Besov estimates as well as some interpolation and time-weighted estimates.

1.1. Notation. Throughout this paper, ∇^k with an integer $k \geqslant 0$ represents the spatial derivatives of order k. When $k < 0$ or k is not a positive integer, ∇^k means Λ^k defined by $\Lambda^k f := \mathscr{F}^{-1}(|\xi|^k \mathscr{F} f)$, where $\mathscr{F}$ is the usual Fourier transform operator and $\mathscr{F}^{-1}$ is its inverse. We denote by $L^p(\mathbb{R}^3)(1 \leqslant p \leqslant \infty)$ the usual Lebesgue spaces with the norm $\|\cdot\|_{L^p}$. For simplicity, we write $\|\cdot\| = \|\cdot\|_{L^2}$. We use $H^k(\mathbb{R}^3)$ for some integer $k \geqslant 0$ to denote the usual Sobolev spaces with the norm $\|\cdot\|_{H^k}$. We use $\dot{H}^s(\mathbb{R}^3)(s \in \mathbb{R})$ to denote the homogeneous Sobolev spaces with the norm $\|\cdot\|_{\dot{H}^s}$ defined by $\|f\|_{\dot{H}^s} := \|\nabla^s f\|$. It is clear for $H^0 = \dot{H}^0 = L^2$.

We introduce the homogeneous Besov spaces. Let $\phi \in C_0^\infty(\mathbb{R}_\xi^3)$ satisfy that $\phi(\xi) = 1$ if $|\xi| \leqslant 1$ and $\phi(\xi) = 0$ if $|\xi| \geqslant 2$. Define $\varphi(\xi) := \phi(\xi) - \phi(2\xi)$ and $\varphi_j(\xi) := \varphi(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Then, $\sum_{j\in\mathbb{Z}} \varphi_j(\xi) = 1$ if $\xi \neq 0$. Define $\dot{\Delta}_j f := \mathscr{F}^{-1}(\varphi_j) * f$. For $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$, we denote by $\dot{B}^s_{p,\infty}(\mathbb{R}^3)$ the homogeneous Besov spaces with the norm $\|\cdot\|_{\dot{B}^s_{p,\infty}}$ defined by $\|f\|_{\dot{B}^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{sj} \|\dot{\Delta}_j f\|_{L^p}$.

The notation $A \lesssim B$ means that $A \leqslant CB$ for a generic positive constant C. We denote $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We use C_0 to denote a positive constant depending additionally on the initial data. For simplicity, we write $\|(A, B)\|_X := \|A\|_X + \|B\|_X$ and $\int f := \int_{\mathbb{R}^3} f\, dx$. The notation $C^k(0, T; X)(k \geqslant 0)$ denotes the space of X-valued k-times continuously differentiable functions on $[0, \mathrm{T}]$.

Our main results are stated in the following.

Theorem 1. *Let $k \geqslant 3$ be an integer. Assume that $(\rho_0 - 1, u_0, \theta_0 - 1) \in H^k$ satisfying*

$$\|(\rho_0 - 1, u_0, \theta_0 - 1)\|_{H^3} < \delta_0, \quad (6)$$

for some small constant $\delta_0 > 0$. Suppose that $g(\rho)$ is a smooth function of ρ satisfying $g(1) = 1$ and $g'(1) = 0$. Then, the Cauchy problem (3)–(4) admits a unique global solution $(\rho, u, \theta)(t)$ such that for all $t \geqslant 0$ and $3 \leqslant \ell \leqslant k$,

$$\begin{aligned} &\|(\rho - 1, u, \theta - 1)(t)\|_{H^\ell} \\ &\quad + \left(\int_0^t \left(\|\nabla\rho(\varsigma)\|^2_{H^{\ell-1}} + \|(u, \theta - 1)(\varsigma)\|^2_{H^\ell}\right) d\varsigma\right)^{1/2} \\ &\leqslant C\|(\rho_0 - 1, u_0, \theta_0 - 1)\|_{H^\ell}, \end{aligned} \quad (7)$$

where $C > 0$ depends only on γ and τ.

Theorem 2. *Under the assumptions of Theorem 1 and $g''(1) = 0$, if further $(\rho_0 - 1, u_0, \theta_0 - 1) \in \dot{H}^{-s}$ for some $s \in [0, 3/2)$ or $(\rho_0 - 1, u_0, \theta_0 - 1) \in \dot{B}^{-s}_{2,\infty}$ for some $s \in (0, 3/2]$, then for all $t \geqslant 0$,*

$$\left\|\nabla^l(\rho - 1)(t)\right\| \leqslant C_0(1 + t)^{-(l+s)/2}, \quad 0 \leqslant l \leqslant k, \quad (8)$$

$$\begin{cases} \left\|\nabla^l(u, \theta - 1)(t)\right\| \leqslant C_0(1 + t)^{-(l+1+s)/2}, & 0 \leqslant l \leqslant k - 1, \\ \left\|\nabla^k(u, \theta - 1)(t)\right\| \leqslant C_0(1 + t)^{-(k+s)/2}. \end{cases} \quad (9)$$

By Lemmas A.1, A.5, and A.6, we easily obtain the following $L^p - L^q$-type decay rates.

Corollary 3. *Under the assumptions of Theorem 2, if $(\rho_0 - 1, u_0, \theta_0 - 1) \in L^p$ for some $p \in [1, 2]$, then for $2 \leqslant q \leqslant \infty$,*

$$\begin{cases} \left\|\nabla^l(\rho - 1)(t)\right\|_{L^q} \leqslant C_0(1 + t)^{-(3/2)((1/p)-(1/q))-(l/2)}, & 0 \leqslant l \leqslant k - 2, \\ \left\|\nabla^l(u, \theta - 1)(t)\right\|_{L^q} \leqslant C_0(1 + t)^{-(3/2)((1/p)-(1/q))-((l+1)/2)}, & 0 \leqslant l \leqslant k - 3, \\ \left\|\nabla^{k-2}(u, \theta - 1)(t)\right\|_{L^q} \leqslant C_0(1 + t)^{-(3/2)((1/p)-(1/q))-((2k-3)/4)-(1/2q)}. \end{cases} \quad (10)$$

Some remarks for Theorems 1 and 2 and Corollary 3 are given in the following.

Remark 4. From Theorem 1, when $k \geqslant 3$, we only require that the H^3 norms of the initial density, velocity, and temperature are small, while the higher-order Sobolev norms can be arbitrarily large.

Remark 5. We claim that the decay rates in Theorem 2 and Corollary 3 are optimal in the sense that they are consistent with those in the linearized case.

Remark 6. By Corollary 3, we prove the optimal $L^p - L^q$-type decay rates without the smallness assumption on the L^p norm of the initial data.

Remark 7. The additional restriction $g''(1) = 0$ in Theorem 2 is necessary to obtain the negative Sobolev or Besov estimates of solutions since the density perturbation is degenerately dissipative.

The rest of this paper is organized as follows. In Section 2, we establish some refined energy estimates (see Lemmas 8–11) which help us to derive important energy estimates with the minimum derivative counts (see Lemma 12). Then, we prove the global solution (Theorem 1) and the decay rates (Theorem 2) in Sections 3 and 4, respectively. In Appendix A, we list some useful lemmas which will be frequently used in the previous sections. The detailed proof of Lemma 15 is given in Appendix B.

2. Energy Estimates

We choose the constant equilibrium state $(1, 0, 1)$. Define the perturbations

$$\begin{aligned} \varrho &= \rho - 1, \\ u &= u - 0, \\ \Theta &= \theta - 1. \end{aligned} \tag{11}$$

Then, the Cauchy problem (3)–(4) is equivalently written as

$$\varrho_t = -u \cdot \nabla\varrho - (1 + \varrho) \operatorname{div} u, \tag{12}$$

$$u_t + \frac{1}{\tau} u + \nabla\Theta = -u \cdot \nabla u - \frac{1 + \Theta}{1 + \varrho} \nabla\varrho, \tag{13}$$

$$\begin{aligned} \frac{1}{\gamma - 1}\Theta_t + \frac{1}{\tau}\Theta = &-\frac{1}{\gamma - 1} u \cdot \nabla\Theta \\ &- (1 + \Theta) \operatorname{div} u + \frac{1}{\tau}[g(\varrho + 1) - 1], \end{aligned} \tag{14}$$

with

$$(\varrho, u, \Theta)|_{t=0} = (\varrho_0, u_0, \Theta_0) := (\rho_0 - 1, u_0, \theta_0 - 1). \tag{15}$$

Next, we will derive the a priori estimates for equations (12)–(14) by assuming that for sufficiently small $\delta > 0$ and some $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} \|(\varrho, u, \Theta)(t)\|_{H^3} < \delta. \tag{16}$$

By Sobolev's inequality, (16) implies

$$\begin{aligned} \frac{1}{2} \leqslant 1 + \varrho \leqslant \frac{3}{2}, \\ \frac{1}{2} \leqslant 1 + \Theta \leqslant \frac{3}{2}. \end{aligned} \tag{17}$$

By Taylor's expansion, we have

$$g(\varrho + 1) = g(1) + g'(1)\varrho + O(\varrho^2) = 1 + O(\varrho^2), \tag{18}$$

where we have used the assumptions that $g(1) = 1$ and $g'(1) = 0$. Thus, we have

$$g(\varrho + 1) - 1 = O(\varrho^2). \tag{19}$$

First, we derive the zero-order energy estimate for (ϱ, u, Θ).

Lemma 8. *Let $\delta \ll 1$. If $\sup_{0\leqslant t\leqslant T}\|(\varrho, u, \Theta)(t)\|_{H^3} < \delta$, then*

$$\frac{d}{dt}\left\|\left(\varrho, u, \frac{1}{\sqrt{\gamma - 1}}\Theta\right)\right\|^2 + C\|(u, \Theta)\|^2 \lesssim \delta\|\nabla(\varrho, \Theta)\|^2. \tag{20}$$

Proof. Multiplying equations (12)–(14) by ϱ, u, and Θ, respectively, summing the resulting identities up, and then integrating over $\mathbb{R}^3$ by parts, by (19), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\left\|\left(\varrho, u, \frac{1}{\sqrt{\gamma - 1}}\Theta\right)\right\|^2 + \frac{1}{\tau}\|u\|^2 + \frac{1}{\tau}\|\Theta\|^2 \\ &= -\int \operatorname{div} (\varrho u)\varrho - \int\left(u \cdot \nabla u + \frac{\Theta - \varrho}{1 + \varrho}\nabla\varrho\right) \\ &\quad \cdot u - \int\left(\Theta \operatorname{div} u + \frac{1}{\gamma - 1} u \cdot \nabla\Theta - \frac{1}{\tau}O(\varrho^2)\right)\Theta. \end{aligned} \tag{21}$$

Now, we estimate (21) term by term. For the term $-\int \operatorname{div} (\varrho u)\varrho$, by integrating by parts, Hölder's, Sobolev's, and Cauchy's inequalities, we obtain

$$-\int \operatorname{div} (\varrho u)\varrho = \int \varrho u \cdot \nabla\varrho \leqslant \|\varrho\|_{L^6}\|u\|_{L^3}\|\nabla\varrho\| \lesssim \delta\|\nabla\varrho\|^2. \tag{22}$$

By Hölder's, Sobolev's, and Cauchy's inequalities and (17), we obtain

$$\begin{aligned} -\int u \cdot \nabla u \cdot u \leqslant \|\nabla u\|_{L^\infty}\|u\|^2 \lesssim \delta\|u\|^2, \\ -\int \frac{\Theta - \varrho}{1 + \varrho}\nabla\varrho \cdot u \leqslant \left\|\frac{\Theta - \varrho}{1 + \varrho}\right\|_{L^\infty}\|\nabla\varrho\|\|u\| \lesssim \delta\left(\|\nabla\varrho\|^2 + \|u\|^2\right). \end{aligned} \tag{23}$$

Similarly, we have

$$-\int\left(\Theta \operatorname{div} u+\frac{1}{\gamma-1} u \cdot \nabla \Theta\right) \Theta \lesssim \delta\left(\|u\|^2+\|\nabla \Theta\|^2\right), \quad (24)$$

$$\int \frac{1}{\tau} O\left(\varrho^2\right) \Theta \lesssim\|\varrho\|_{L^3}\|\varrho\|_{L^6}\|\Theta\| \lesssim \delta\left(\|\nabla \varrho\|^2+\|\Theta\|^2\right). \quad (25)$$

Plugging the estimates (22)–(25) into (21), since $\delta \ll 1$, we deduce (20). □

Next, we construct the higher-order energy estimates for (ϱ, u, Θ), which include the dissipation estimates for u and Θ of order k.

Lemma 9. *Let $k \geqslant 3$ and $\delta \ll 1$. If $\sup_{0 \leqslant t \leqslant T}\|(\varrho, u, \Theta)(t)\|_{H^3} < \delta$, then for $3 \leqslant \ell \leqslant k$,*

$$\frac{d}{d t} \int \mathfrak{E}^{\ell}(t)+C\left\|\nabla^{\ell}(u, \Theta)\right\|^2 \lesssim \delta\left\|\nabla^{\ell} \varrho\right\|^2, \quad (26)$$

where

$$\begin{aligned} \mathfrak{E}^{\ell}(t):= & \frac{1+\Theta}{1+\varrho}\left|\nabla^{\ell} \varrho\right|^2+(1+\varrho)\left|\nabla^{\ell-1} \operatorname{div} u\right|^2 \\ & +\left|\nabla^{\ell-1} \operatorname{curl} u\right|^2+\frac{1}{\gamma-1} \frac{1+\varrho}{1+\Theta}\left|\nabla^{\ell} \Theta\right|^2 . \end{aligned} \quad (27)$$

Proof. For equations (12)–(14), computing

$$\begin{aligned} \int\Big[& \frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot \nabla^{\ell}(\text{equation }(12)) \\ & +(1+\varrho) \nabla^{\ell-1} \operatorname{div} u \cdot \nabla^{\ell-1} \operatorname{div}(\text{equation }(13)) \\ & +\frac{1+\varrho}{1+\Theta} \nabla^{\ell} \Theta \cdot \nabla^{\ell}(\text{equation }(14))\Big], \end{aligned} \quad (28)$$

by (19), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{d t} \int\left[\frac{1+\Theta}{1+\varrho}\left|\nabla^{\ell} \varrho\right|^2+(1+\varrho)\left|\nabla^{\ell-1} \operatorname{div} u\right|^2\right. \\ & \left.+\frac{1}{\gamma-1} \frac{1+\varrho}{1+\Theta}\left|\nabla^{\ell} \Theta\right|^2\right]+\frac{1}{\tau} \int\left[(1+\varrho)\left|\nabla^{\ell-1} \operatorname{div} u\right|^2\right. \\ & \left.+\frac{1+\varrho}{1+\Theta}\left|\nabla^{\ell} \Theta\right|^2\right]=\frac{1}{2} \int\left[\partial_t\left(\frac{1+\Theta}{1+\varrho}\right)\left|\nabla^{\ell} \varrho\right|^2+\partial_t \varrho\left|\nabla^{\ell-1} \operatorname{div} u\right|^2\right. \\ & \left.+\frac{1}{\gamma-1} \partial_t\left(\frac{1+\varrho}{1+\Theta}\right)\left|\nabla^{\ell} \Theta\right|^2\right]-\int\left\{\frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot \nabla^{\ell}[(1+\varrho) \operatorname{div} u]\right. \\ & \left.+(1+\varrho) \nabla^{\ell-1} \operatorname{div} u \cdot \nabla^{\ell-1} \operatorname{div}\left(\frac{1+\Theta}{1+\varrho} \nabla \varrho\right)\right\} \\ & -\int\left\{(1+\varrho) \nabla^{\ell-1} \operatorname{div} u \cdot \nabla^{\ell-1} \Delta \Theta+\frac{1+\varrho}{1+\Theta} \nabla^{\ell} \Theta \cdot \nabla^{\ell}[(1+\Theta) \operatorname{div} u]\right\} \\ & -\int \frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot \nabla^{\ell}(u \cdot \nabla \varrho)-\int(1+\varrho) \nabla^{\ell-1} \operatorname{div} u \cdot \nabla^{\ell-1} \operatorname{div}(u \cdot \nabla u) \\ & -\frac{1}{\gamma-1} \int \frac{1+\varrho}{1+\Theta} \nabla^{\ell} \Theta \cdot \nabla^{\ell}(u \cdot \nabla \Theta)+\frac{1}{\tau} \int \frac{1+\varrho}{1+\Theta} \nabla^{\ell} \Theta \cdot \nabla^{\ell} O\left(\varrho^2\right):=\sum_{i=1}^7 I_i . \end{aligned} \quad (29)$$

Now, we estimate the terms I_1–I_7. By (12), (14), and (17) and Hölder's and Sobolev's inequalities, we have

$$\begin{aligned} I_1 & \lesssim\left(\left\|\Theta_t\right\|_{L^{\infty}}+\left\|\varrho_t\right\|_{L^{\infty}}\right)\left\|\nabla^{\ell}(\varrho, u, \Theta)\right\|^2 \\ & \lesssim\|(\varrho, u, \Theta)\|_{H^3}\left\|\nabla^{\ell}(\varrho, u, \Theta)\right\|^2 \lesssim \delta\left\|\nabla^{\ell}(\varrho, u, \Theta)\right\|^2 . \end{aligned} \quad (30)$$

By the commutator notation (A.3), the commutator estimates (A.4), (17), integrating by parts, and Lemma A.3, we have

$$\begin{aligned} I_2= & -\int\left[(1+\Theta) \nabla^{\ell} \varrho \cdot \nabla^{\ell} \operatorname{div} u+(1+\Theta) \nabla^{\ell-1} \operatorname{div} u \cdot \nabla^{\ell-1} \Delta \varrho\right] \\ & -\int\left\{\frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot\left[\nabla^{\ell}, 1+\varrho\right] \operatorname{div} u+(1+\varrho) \nabla^{\ell-1} \operatorname{div} u\right. \\ & \left.\cdot\left[\nabla^{\ell-1} \operatorname{div}, \frac{1+\Theta}{1+\varrho}\right] \nabla \varrho\right\} \lesssim \int|\nabla \Theta|\left|\nabla^{\ell} \varrho\right|\left|\nabla^{\ell} u\right| \\ & +\int\left|\nabla^{\ell} \varrho\right|\left|\left[\nabla^{\ell}, 1+\varrho\right] \nabla u\right|+\int\left|\nabla^{\ell} u\right|\left|\left[\nabla^{\ell}, \frac{1+\Theta}{1+\varrho}\right] \nabla \varrho\right| \\ \lesssim & \|\nabla \Theta\|_{L^{\infty}}\left\|\nabla^{\ell} \varrho\right\|\left\|\nabla^{\ell} u\right\|+\left\|\nabla^{\ell} \varrho\right\|\left\|\left[\nabla^{\ell}, 1+\varrho\right] \nabla u\right\| \\ & +\left\|\nabla^{\ell} u\right\|\left\|\left[\nabla^{\ell}, \frac{1+\Theta}{1+\varrho}\right] \nabla \varrho\right\| \lesssim\|\nabla \Theta\|_{L^{\infty}}\left\|\nabla^{\ell} u\right\|\left\|\nabla^{\ell} \varrho\right\| \\ & +\left\|\nabla^{\ell} \varrho\right\|\left(\|\nabla \varrho\|_{L^{\infty}}\left\|\nabla^{\ell} u\right\|+\left\|\nabla^{\ell} \varrho\right\|\|\nabla u\|_{L^{\infty}}\right)+\left\|\nabla^{\ell} u\right\| \\ & \cdot\left(\left\|\nabla\left(\frac{1+\Theta}{1+\varrho}\right)\right\|_{L^{\infty}}\left\|\nabla^{\ell} \varrho\right\|+\left\|\nabla^{\ell}\left(\frac{1+\Theta}{1+\varrho}\right)\right\|\|\nabla \varrho\|_{L^{\infty}}\right) \\ \lesssim & \delta\left\|\nabla^{\ell}(\varrho, u, \Theta)\right\|^2, \\ I_3= & -\int(1+\varrho) \nabla^{\ell-1} \operatorname{div} u \cdot \nabla^{\ell-1} \Delta \Theta+(1+\varrho) \nabla^{\ell} \Theta \cdot \nabla^{\ell} \operatorname{div} u \\ & -\int \frac{1+\varrho}{1+\Theta} \nabla^{\ell} \Theta \cdot\left[\nabla^{\ell}, 1+\Theta\right] \operatorname{div} u \lesssim \int|\nabla \varrho|\left|\nabla^{\ell} \Theta\right|\left|\nabla^{\ell-1} \operatorname{div} u\right| \\ & +\int\left|\nabla^{\ell} \Theta\right|\left|\left[\nabla^{\ell}, 1+\Theta\right] \nabla u\right| \lesssim\|\nabla \varrho\|_{L^{\infty}}\left\|\nabla^{\ell} \Theta\right\|\left\|\nabla^{\ell} u\right\| \\ & +\left\|\nabla^{\ell} \Theta\right\|\left(\|\nabla \Theta\|_{L^{\infty}}\left\|\nabla^{\ell} u\right\|+\left\|\nabla^{\ell} \Theta\right\|\|\nabla u\|_{L^{\infty}}\right) \\ \lesssim & \delta\left\|\nabla^{\ell}(u, \Theta)\right\|^2 . \end{aligned} \quad (31)$$

By (17), Lemma A.2, integrating by parts, and Hölder's, Sobolev's, and Cauchy's inequalities, we have

$$\begin{aligned} I_4= & -\int \frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot \nabla^{\ell}(u \cdot \nabla \varrho)=-\frac{1}{2} \int \frac{1+\Theta}{1+\varrho} u \cdot \nabla\left|\nabla^{\ell} \varrho\right|^2 \\ & -\int \frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot\left[\nabla^{\ell}, u\right] \cdot \nabla \varrho=\frac{1}{2} \int \operatorname{div}\left(\frac{1+\Theta}{1+\varrho} u\right)\left|\nabla^{\ell} \varrho\right|^2 \\ & -\int \frac{1+\Theta}{1+\varrho} \nabla^{\ell} \varrho \cdot\left[\nabla^{\ell}, u\right] \cdot \nabla \varrho \lesssim \delta\left\|\nabla^{\ell} \varrho\right\|^2+\left\|\nabla^{\ell} \varrho\right\|\left\|\left[\nabla^{\ell}, u\right]\right. \\ & \left.\cdot \nabla \varrho\right\| \lesssim \delta\left\|\nabla^{\ell} \varrho\right\|^2+\left\|\nabla^{\ell} \varrho\right\|\left(\|\nabla u\|_{L^{\infty}}\left\|\nabla^{\ell} \varrho\right\|\right. \\ & \left.+\left\|\nabla^{\ell} u\right\|\|\nabla \varrho\|_{L^{\infty}}\right) \lesssim \delta\left\|\nabla^{\ell}(\varrho, u)\right\|^2 . \end{aligned} \quad (32)$$

Note that the vector formula

$$\operatorname{div}(u \cdot \nabla u)=\nabla u:(\nabla u)^T+u \cdot \nabla \operatorname{div} u, \quad (33)$$

where the double dots : means that $\mathbb{A}: \mathbb{B}=\sum_{i, j=1}^3 a_{i j} b_{i j}$ for two

3×3 matrices $\mathbb{A}=(a_{ij})$ and $\mathbb{B}=(b_{ij})$. By integrating by parts, (17) and (33), Hölder's, Sobolev's, and Cauchy's inequalities, and Lemma A.2, we estimate

$$
\begin{aligned}
I_5 &= -\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\nabla^{\ell-1}\operatorname{div}(u\cdot\nabla u)\\
&= -\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\nabla^{\ell-1}\left[\nabla u:(\nabla u)^T+u\cdot\nabla\operatorname{div} u\right]\\
&= -\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\nabla^{\ell-1}\left[\nabla u:(\nabla u)^T\right]\\
&\quad -\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\nabla^{\ell-1}(u\cdot\nabla\operatorname{div} u)\\
&= -\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\nabla^{\ell-1}\left[\nabla u:(\nabla u)^T\right]\\
&\quad -\frac{1}{2}\int (1+\varrho)u\cdot\nabla\left|\nabla^{\ell-1}\operatorname{div} u\right|^2-\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\left[\nabla^{\ell-1},u\right]\cdot\nabla\operatorname{div} u\\
&= -\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\nabla^{\ell-1}\left[\nabla u:(\nabla u)^T\right]\\
&\quad +\frac{1}{2}\int \operatorname{div}\left[(1+\varrho)u\right]\left|\nabla^{\ell-1}\operatorname{div} u\right|^2-\int (1+\varrho)\nabla^{\ell-1}\operatorname{div} u\cdot\left[\nabla^{\ell-1},u\right]\cdot\nabla\operatorname{div} u\\
&\lesssim \left\|\nabla^{\ell-1}\operatorname{div} u\right\|\left\|\nabla^{\ell-1}\left[\nabla u:(\nabla u)^T\right]\right\|+\left\|\operatorname{div}\left[(1+\varrho)u\right]\right\|_{L^\infty}\left\|\nabla^{\ell-1}\operatorname{div} u\right\|^2\\
&\quad +\left\|\nabla^{\ell-1}\operatorname{div} u\right\|\left\|\left[\nabla^{\ell-1},u\right]\cdot\nabla\operatorname{div} u\right\|\lesssim\left\|\nabla^{\ell}u\right\|\left\|\nabla u\right\|_{L^\infty}\left\|\nabla^{\ell}u\right\|+\delta\left\|\nabla^{\ell}u\right\|^2\\
&\quad +\left\|\nabla^{\ell}u\right\|\left(\left\|\nabla^{\ell-1}u\right\|_{L^6}\left\|\nabla\operatorname{div} u\right\|_{L^3}+\left\|\nabla u\right\|_{L^\infty}\left\|\nabla^{\ell-1}\operatorname{div} u\right\|\right)\lesssim\delta\left\|\nabla^{\ell}u\right\|^2,\\
I_6 &= -\frac{1}{\gamma-1}\int\frac{1+\varrho}{1+\Theta}\nabla^{\ell}\Theta\cdot\nabla^{\ell}(u\cdot\nabla\Theta)\\
&= -\frac{1}{\gamma-1}\int\frac{1+\varrho}{1+\Theta}\nabla^{\ell}\Theta\cdot u\cdot\nabla\nabla^{\ell}\Theta-\frac{1}{\gamma-1}\int\frac{1+\varrho}{1+\Theta}\nabla^{\ell}\Theta\cdot\left[\nabla^{\ell},u\right]\cdot\nabla\Theta\\
&= \frac{1}{2}\frac{1}{\gamma-1}\int\operatorname{div}\left(\frac{1+\varrho}{1+\Theta}u\right)\left|\nabla^{\ell}\Theta\right|^2-\frac{1}{\gamma-1}\int\frac{1+\varrho}{1+\Theta}\nabla^{\ell}\Theta\cdot\left[\nabla^{\ell},u\right]\cdot\nabla\Theta\\
&\lesssim\left\|\operatorname{div}\left(\frac{1+\varrho}{1+\Theta}u\right)\right\|_{L^\infty}\left\|\nabla^{\ell}\Theta\right\|^2\\
&\quad +\left\|\frac{1+\varrho}{1+\Theta}\right\|_{L^\infty}\left\|\nabla^{\ell}\Theta\right\|\left\|\left[\nabla^{\ell},u\right]\cdot\nabla\Theta\right\|\lesssim\delta\left\|\nabla^{\ell}\Theta\right\|^2\\
&\quad +\left\|\nabla^{\ell}\Theta\right\|\left(\left\|\nabla^{\ell}u\right\|\left\|\nabla\Theta\right\|_{L^\infty}+\left\|\nabla u\right\|_{L^\infty}\left\|\nabla^{\ell-1}\nabla\Theta\right\|\right)\lesssim\delta\left\|\nabla^{\ell}(u,\Theta)\right\|^2.
\end{aligned}\tag{34}
$$

By (17), the product estimates (A.5) of Lemma A.2, and Corollary A.4, we have

$$
\begin{aligned}
I_7 &= \frac{1}{\tau}\int\frac{1+\varrho}{1+\Theta}\nabla^{\ell}\Theta\cdot\nabla^{\ell}O(\varrho^2)\lesssim\int\left|\nabla^{\ell}\Theta\right|\left|\nabla^{\ell}\varrho\right|\left|O(\varrho)\right|\\
&\lesssim\delta\left\|\nabla^{\ell}(\varrho,\Theta)\right\|^2.
\end{aligned}\tag{35}
$$

Plugging the estimates for I_1–I_7 into (29), by (17), since δ is small, we deduce

$$
\begin{aligned}
&\frac{d}{dt}\int\left[\frac{1+\Theta}{1+\varrho}\left|\nabla^{\ell}\varrho\right|^2+(1+\varrho)\left|\nabla^{\ell-1}\operatorname{div} u\right|^2+\frac{1+\varrho}{1+\Theta}\left|\nabla^{\ell}\Theta\right|^2\right]\\
&\quad +C\left(\left\|\nabla^{\ell-1}\operatorname{div} u\right\|^2+\left\|\nabla^{\ell}\Theta\right\|^2\right)\lesssim\delta\left\|\nabla^{\ell}(\varrho,u)\right\|^2.
\end{aligned}\tag{36}
$$

Rewrite equation (13) as

$$
u_t+\frac{1}{\tau}u=-\nabla\varrho-\nabla\Theta-u\cdot\nabla u+\frac{\varrho-\Theta}{1+\varrho}\nabla\varrho.\tag{37}
$$

Applying curl to (37), we obtain

$$
(\operatorname{curl} u)_t+\frac{1}{\tau}\operatorname{curl} u=-\operatorname{curl}(u\cdot\nabla u)+\nabla\left(\frac{\varrho-\Theta}{1+\varrho}\right)\times\nabla\varrho,\tag{38}
$$

where $\times$ represents the cross product of vectors. Applying $\nabla^{\ell-1}$ to (38), multiplying the resulting identity by $\nabla^{\ell-1}\operatorname{curl} u$, and integrating over $\mathbb{R}^3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left|\nabla^{\ell-1}\operatorname{curl} u\right|^2+\frac{1}{\tau}\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|^2\\
&\quad=-\int\nabla^{\ell-1}\operatorname{curl}(u\cdot\nabla u)\cdot\nabla^{\ell-1}\operatorname{curl} u\\
&\qquad+\int\nabla^{\ell-1}\left[\nabla\left(\frac{\varrho-\Theta}{1+\varrho}\right)\times\nabla\varrho\right]\cdot\nabla^{\ell-1}\operatorname{curl} u:=I_8+I_9.
\end{aligned}\tag{39}
$$

By (17), integrating by parts, Hölder's, Sobolev's, and Cauchy's inequalities, and Lemmas A.2 and A.3, we estimate

$$
\begin{aligned}
I_8 &= -\int\nabla^{\ell-1}\operatorname{curl}(u\cdot\nabla u)\cdot\nabla^{\ell-1}\operatorname{curl} u=-\frac{1}{2}\int u\\
&\quad\cdot\nabla\left|\nabla^{\ell-1}\operatorname{curl} u\right|^2-\int\left[\nabla^{\ell-1}\operatorname{curl},u\right]\cdot\nabla u\cdot\nabla^{\ell-1}\operatorname{curl} u\\
&= \frac{1}{2}\int\operatorname{div} u\left|\nabla^{\ell-1}\operatorname{curl} u\right|^2-\int\left[\nabla^{\ell-1}\operatorname{curl},u\right]\\
&\quad\cdot\nabla u\cdot\nabla^{\ell-1}\operatorname{curl} u\lesssim\delta\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|^2\\
&\quad+\int\left|\left[\nabla^{\ell},u\right]\cdot\nabla u\right|\left|\nabla^{\ell}u\right|\lesssim\delta\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|^2\\
&\quad+\left\|\left[\nabla^{\ell},u\right]\cdot\nabla u\right\|\left\|\nabla^{\ell}u\right\|\lesssim\delta\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|^2\\
&\quad+\left\|\nabla u\right\|_{L^\infty}\left\|\nabla^{\ell}u\right\|^2\lesssim\delta\left\|\nabla^{\ell}u\right\|^2,\\
I_9 &= \int\nabla^{\ell-1}\left[\nabla\left(\frac{\varrho-\Theta}{1+\varrho}\right)\times\nabla\varrho\right]\\
&\quad\cdot\nabla^{\ell-1}\operatorname{curl} u\left\|\nabla^{\ell-1}\left[\nabla\left(\frac{\varrho-\Theta}{1+\varrho}\right)\times\nabla\varrho\right]\right\|\\
&\quad\cdot\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|\lesssim\left(\left\|\nabla^{\ell}\left(\frac{\varrho-\Theta}{1+\varrho}\right)\right\|\left\|\nabla\varrho\right\|_{L^\infty}\right.\\
&\quad\left.+\left\|\nabla\left(\frac{\varrho-\Theta}{1+\varrho}\right)\right\|_{L^\infty}\left\|\nabla^{\ell}\varrho\right\|\right)\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|\\
&\lesssim\left(\left\|\nabla^{\ell}(\varrho,\Theta)\right\|\left\|\varrho\right\|_{H^3}+\left\|(\varrho,\Theta)\right\|_{H^3}\left\|\nabla^{\ell}\varrho\right\|\right)\\
&\quad\cdot\left\|\nabla^{\ell}u\right\|\lesssim\delta\left\|\nabla^{\ell}(\varrho,u,\Theta)\right\|^2.
\end{aligned}\tag{40}
$$

Plugging the estimates for I_8 and I_9 into (39), we obtain

$$
\frac{d}{dt}\int\left|\nabla^{\ell-1}\operatorname{curl} u\right|^2+C\left\|\nabla^{\ell-1}\operatorname{curl} u\right\|^2\lesssim\delta\left\|\nabla^{\ell}(\varrho,u,\Theta)\right\|^2.\tag{41}
$$

Adding (41) to (36), noting

$$\left\|\nabla^{\ell} u\right\|^2=\left\|\nabla^{\ell-1} \operatorname{div} u\right\|^2+\left\|\nabla^{\ell-1} \operatorname{curl} u\right\|^2, \quad (42)$$

since $\delta \ll 1$, we deduce (26). □

Corollary 10. *Let $k \geqslant 3$ and $\delta \ll 1$. If $\sup_{0 \leqslant t \leqslant T}\|(\varrho, u, \Theta)(t)\|_{H^3}<\delta$, then for $1 \leqslant \ell \leqslant k-1$,*

$$\frac{d}{dt}\left\|\nabla^{\ell}\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}} \Theta\right)\right\|^2+C\left\|\nabla^{\ell}(u, \Theta)\right\|^2 \lesssim \delta\left\|\nabla^{\ell+1}(\varrho, \Theta)\right\|^2. \quad (43)$$

Combining with Lemma 8, we have for $0 \leqslant \ell \leqslant k-1$,

$$\frac{d}{dt}\left\|\nabla^{\ell}\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}} \Theta\right)\right\|^2+C\left\|\nabla^{\ell}(u, \Theta)\right\|^2 \lesssim \delta\left\|\nabla^{\ell+1}(\varrho, \Theta)\right\|^2. \quad (44)$$

Proof. Applying ∇^{ℓ} to equations (12)–(14), then multiplying the resulting identities by $\nabla^{\ell}\varrho$, $\nabla^{\ell}u$, and $\nabla^{\ell}\Theta$, respectively, summing them up, and then integrating over $\mathbb{R}^3$ by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\nabla^{\ell}\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}} \Theta\right)\right\|^2+\frac{1}{\tau}\left\|\nabla^{\ell}(u, \Theta)\right\|^2 \\
&=-\int \nabla^{\ell} \operatorname{div}(\varrho u) \cdot \nabla^{\ell} \varrho-\int\left[\nabla^{\ell}(u \cdot \nabla u) \cdot \nabla^{\ell} u+\nabla^{\ell}\left(\frac{\Theta-\varrho}{1+\varrho} \nabla \varrho\right) \cdot \nabla^{\ell} u\right] \\
&-\int\left[\frac{1}{\gamma-1} \nabla^{\ell}(u \cdot \nabla \Theta) \cdot \nabla^{\ell} \Theta+\nabla^{\ell}(\Theta \operatorname{div} u) \cdot \nabla^{\ell} \Theta-\frac{1}{\tau} \nabla^{\ell}\left(O\left(\varrho^2\right)\right) \cdot \nabla^{\ell} \Theta\right].
\end{aligned} \quad (45)$$

Then, as in the proof of Lemma 9, we easily deduce (43) from (45). □

We shall derive the dissipation estimates for ϱ up to order k.

Lemma 11. *Let $k \geqslant 3$ and $\delta \ll 1$. If $\sup_{0 \leqslant t \leqslant T}\|(\varrho, u, \Theta)(t)\|_{H^3}<\delta$, then for $0 \leqslant \ell \leqslant k-1$,*

$$\frac{d}{dt} \int \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho+\left\|\nabla^{\ell+1} \varrho\right\|^2 \lesssim\left\|\nabla^{\ell} u\right\|^2+\left\|\nabla^{\ell+1}(u, \Theta)\right\|^2. \quad (46)$$

Proof. Rewrite equation (13) as

$$\nabla \varrho=-u_t-\frac{1}{\tau} u-\nabla \Theta-u \cdot \nabla u+\frac{\varrho-\Theta}{1+\varrho} \nabla \varrho. \quad (47)$$

Applying ∇^{ℓ} to (47), multiplying the resulting identity by $\nabla\nabla^{\ell}\varrho$, and then integrating over $\mathbb{R}^3$, by Hölder's and Cauchy's inequalities, we have

$$\begin{aligned}
\left\|\nabla^{\ell+1} \varrho\right\|^2 \lesssim & -\int \partial_t \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho+\left\|\nabla^{\ell} u\right\|^2+\left\|\nabla^{\ell+1} \Theta\right\|^2 \\
& +\left\|\nabla^{\ell}(u \cdot \nabla u)\right\|^2+\left\|\nabla^{\ell}\left(\frac{\varrho-\Theta}{1+\varrho} \nabla \varrho\right)\right\|^2 .
\end{aligned} \quad (48)$$

By (12), we integrate by parts to obtain

$$\begin{aligned}
-\int \partial_t \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho & =-\frac{d}{dt} \int \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho+\int \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho_t \\
& =-\frac{d}{dt} \int \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho+\int \nabla^{\ell} \operatorname{div} u \cdot\left[\nabla^{\ell} \operatorname{div} u+\nabla^{\ell} \operatorname{div}(\varrho u)\right] \\
& \lesssim-\frac{d}{dt} \int \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho+\|\nabla^{\ell+1} u\|^2+\|\nabla^{\ell+1}(\varrho u)\|^2 \\
& \lesssim-\frac{d}{dt} \int \nabla^{\ell} u \cdot \nabla \nabla^{\ell} \varrho+\left\|\nabla^{\ell+1} u\right\|^2+\delta\left\|\nabla^{\ell+1}(\varrho, u)\right\|^2,
\end{aligned} \quad (49)$$

where we have used the product estimates (A.5) of Lemma A.2 to estimate

$$\left\|\nabla^{\ell+1}(\varrho u)\right\| \lesssim\|\varrho\|_{L^{\infty}}\left\|\nabla^{\ell+1} u\right\|+\left\|\nabla^{\ell+1} \varrho\right\|\|u\|_{L^{\infty}} \lesssim \delta\left\|\nabla^{\ell+1}(\varrho, u)\right\|. \quad (50)$$

By Lemmas A.2 and A.3, we have

$$\left\|\nabla^{\ell}(u \cdot \nabla u)\right\| \lesssim\|u\|_{L^{\infty}}\left\|\nabla^{\ell+1} u\right\|+\left\|\nabla^{\ell} u\right\|_{L^6}\|\nabla u\|_{L^3} \lesssim \delta\left\|\nabla^{\ell+1} u\right\|, \quad (51)$$

$$\begin{aligned}
\left\|\nabla^{\ell}\left(\frac{\varrho-\Theta}{1+\varrho} \nabla \varrho\right)\right\| & \lesssim\|(\varrho, \Theta)\|_{L^{\infty}}\left\|\nabla^{\ell+1} \varrho\right\| \\
& +\left\|\nabla^{\ell}\left(\frac{\varrho-\Theta}{1+\varrho}\right)\right\|_{L^6}\|\nabla \varrho\|_{L^3} \lesssim \delta\left\|\nabla^{\ell+1}(\varrho, \Theta)\right\|.
\end{aligned} \quad (52)$$

Plugging (49)–(52) into (48), we deduce (46). □

Finally, we collect all the dissipation estimates for (ϱ, u, Θ) in Lemma 9, Corollary 10, and Lemma 11 to derive the following energy inequality with the minimum derivative counts.

Lemma 12. *Let $k \geqslant 3$ and $T>0$. If $\sup_{0 \leqslant t \leqslant T}\|(\varrho, u, \Theta)(t)\|_{H^3}<\delta \ll 1$, then there exists an energy functional $\mathscr{E}_l^k(t)$, which is equivalent to $\|\nabla^l(\varrho, u, \Theta)(t)\|_{H^{k-l}}^2$, such that for any $t \in[0, T]$ and $0 \leqslant l \leqslant k-1$,*

$$\frac{d}{dt} \mathscr{E}_l^k(t)+\left\|\nabla^{l+1} \varrho(t)\right\|_{H^{k-l-1}}^2+\left\|\nabla^l(u, \Theta)(t)\right\|_{H^{k-l}}^2 \leqslant 0. \quad (53)$$

Proof. Let $k \geqslant 3$ and $0 \leqslant l \leqslant k-1$. Summing up (44) of Corollary 10 from $\ell=l$ to $\ell=k-1$ and adding the resulting identity to (26) of Lemma 9 which $\ell = k$, since δ is small, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\left\|\nabla^l\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}}\Theta\right)(t)\right\|^2_{H^{k-l-1}} + \int \mathfrak{E}^k(t)\right] \\ &\quad + C_1\left\|\nabla^l(u,\Theta)(t)\right\|^2_{H^{k-l}} \leqslant C_2\delta\left\|\nabla^{l+1}\varrho(t)\right\|^2_{H^{k-l-1}}. \end{aligned} \tag{54}$$

Summing up (46) of Lemma 11 from $\ell = l$ to $\ell = k-1$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int\sum_{\ell=l}^{k-1}\nabla^\ell u\cdot\nabla\nabla^\ell\varrho + C_3\left\|\nabla^{l+1}\varrho(t)\right\|^2_{H^{k-l-1}} \\ &\quad \leqslant C_4\left(\left\|\nabla^l u(t)\right\|^2_{H^{k-l}} + \left\|\nabla^{l+1}\Theta(t)\right\|^2_{H^{k-l-1}}\right). \end{aligned} \tag{55}$$

Multiplying (55) by $(2C_2\delta)/C_3$ and then adding it to (54), since $\delta \ll 1$, we deduce

$$\begin{aligned} &\frac{d}{dt}\left[\left\|\nabla^l\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}}\Theta\right)(t)\right\|^2_{H^{k-l-1}} + \int \mathfrak{E}^k(t)\right. \\ &\quad \left. + \frac{2C_2\delta}{C_3}\int\sum_{\ell=l}^{k-1}\nabla^\ell u\cdot\nabla\nabla^\ell\varrho\right] \\ &\quad + C_5\left(\left\|\nabla^{l+1}\varrho(t)\right\|^2_{H^{k-l-1}} + \left\|\nabla^l(u,\Theta)(t)\right\|^2_{H^{k-l}}\right) \leqslant 0. \end{aligned} \tag{56}$$

We define

$$\begin{aligned} \mathscr{E}_l^k(t) &:= \frac{1}{C_5}\left[\left\|\nabla^l\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}}\Theta\right)(t)\right\|^2_{H^{k-l-1}}\right. \\ &\quad \left. + \int \mathfrak{E}^k(t) + \frac{2C_2\delta}{C_3}\int\sum_{\ell=l}^{k-1}\nabla^\ell u\cdot\nabla\nabla^\ell\varrho\right]. \end{aligned} \tag{57}$$

Note that

$$\begin{aligned} \mathfrak{E}^k(t) &= \frac{1+\Theta}{1+\varrho}\left|\nabla^k\varrho\right|^2 + (1+\varrho)\left|\nabla^{k-1}\operatorname{div} u\right|^2 \\ &\quad + \left|\nabla^{k-1}\operatorname{curl} u\right|^2 + \frac{1}{\gamma-1}\frac{1+\varrho}{1+\Theta}\left|\nabla^k\Theta\right|^2. \end{aligned} \tag{58}$$

By (17) and (42), since $\delta \ll 1$, we can deduce from (57) and (58) that there exists a positive constant c such that for any $t \in [0, T]$,

$$\frac{1}{c}\left\|\nabla^l(\varrho,u,\Theta)(t)\right\|^2_{H^{k-l}} \leqslant \mathscr{E}_l^k(t) \leqslant c\left\|\nabla^l(\varrho,u,\Theta)(t)\right\|^2_{H^{k-l}}. \tag{59}$$

Hence, the proof of Lemma 12 is completed. □

3. Global Solution

In this section, we will prove the existence and uniqueness of the global solution, namely, Theorem 1. We first record the local solution (cf. [38]).

Proposition 13 (local-in-time solution). *Assume that* $(\varrho_0, u_0, \Theta_0) \in H^3$ *and* $\inf_{x\in\mathbb{R}^3}\{\varrho_0(x)+1\} > 0$. *Then, there exists a constant* $T > 0$ *such that the Cauchy problem (12)–(15) admits a unique solution* $(\varrho, u, \Theta)(t) \in \mathscr{E}(0, T; H^3)$ *satisfying*

$$\begin{cases} \inf\limits_{x\in\mathbb{R}^3, 0\leqslant t\leqslant T}\{\varrho(x,t)+1\} > 0, \\ \sup\limits_{0\leqslant t\leqslant T}\|(\varrho,u,\Theta)(t)\|_{H^3} \leqslant C_1\|(\varrho_0,u_0,\Theta_0)\|_{H^3}, \end{cases} \tag{60}$$

where $C_1 > 1$ *is some fixed constant. Here,*

$$\begin{aligned} \mathscr{E}\left(0,T;H^3\right) &:= \left\{(\varrho,u,\Theta)(x,t) : (\varrho,u,\Theta)(x,t) \in C^0\right. \\ &\quad \left.\cdot\left(0,T;H^3\right)\cap C^1\left(0,T;H^2\right)\right\}. \end{aligned} \tag{61}$$

Then, we construct the a priori estimates by using the energy estimates given in Lemma 12.

Proposition 14 (a priori estimates). *Let* $k \geqslant 3$ *and* $T > 0$. *Assume that for some sufficiently small* $\delta > 0$,

$$\sup_{0\leqslant t\leqslant T}\|(\varrho,u,\Theta)(t)\|_{H^3} < \delta. \tag{62}$$

Then, we have for any $t \in [0, T]$ *and* $3 \leqslant \ell \leqslant k$,

$$\begin{aligned} &\|(\varrho,u,\Theta)(t)\|_{H^\ell} + \left(\int_0^t\left(\|\nabla\varrho(\varsigma)\|^2_{H^{\ell-1}} + \|(u,\Theta)(\varsigma)\|^2_{H^\ell}\right)d\varsigma\right)^{1/2} \\ &\quad \leqslant C_2\|(\varrho_0,u_0,\Theta_0)\|_{H^\ell}, \end{aligned} \tag{63}$$

where $C_2 > 1$ *is some fixed constant.*

Proof. Let $k \geqslant 3$ and $3 \leqslant \ell \leqslant k$. Letting $l = 0$ and $k = \ell$ in (53) of Lemma 12, we obtain

$$\frac{d}{dt}\mathscr{E}_0^\ell(t) + \|\nabla\varrho(t)\|^2_{H^{\ell-1}} + \|(u,\Theta)(t)\|^2_{H^\ell} \leqslant 0. \tag{64}$$

Integrating (64) in time, we obtain

$$\mathscr{E}_0^\ell(t) + \int_0^t\left(\|\nabla\varrho(\varsigma)\|^2_{H^{\ell-1}} + \|(u,\Theta)(\varsigma)\|^2_{H^\ell}\right)d\varsigma \leqslant \mathscr{E}_0^\ell(0). \tag{65}$$

Letting $l = 0$ and $k = \ell$ in (59), we obtain for any $t \in [0, T]$ and some $c > 0$,

$$\frac{1}{c}\|(\varrho,u,\Theta)(t)\|^2_{H^\ell} \leqslant \mathscr{E}_0^\ell(t) \leqslant c\|(\varrho,u,\Theta)(t)\|^2_{H^\ell}. \tag{66}$$

We immediately deduce (63) from (65) and (66). □

Finally, we perform a continuous argument to extend the local solution given in Proposition 13 to the global one.

Let $k \geqslant 3$. Assume $(\varrho_0, u_0, \Theta_0) \in H^k$ satisfying

$$\|(\varrho_0, u_0, \Theta_0)\|_{H^3} < \frac{\delta}{C_1 C_2}, \tag{67}$$

where $C_1 > 1$, $C_2 > 1$, and $\delta > 0$ are given by Propositions 13 and 14. Since

$$\|(\varrho_0, u_0, \Theta_0)\|_{H^3} < \frac{\delta}{C_1}, \tag{68}$$

by Proposition 13, there exists a constant $T_1 > 0$ such that the Cauchy problem (12)–(15) has a unique local solution

$$(\varrho, u, \Theta)(t) \in \mathscr{E}\left(0, T_1 ; H^3\right), \tag{69}$$

which satisfies

$$\|(\varrho, u, \Theta)(t)\|_{H^3} \leqslant C_1 \|(\varrho_0, u_0, \Theta_0)\|_{H^3} < \delta, \quad \forall t \in [0, T_1]. \tag{70}$$

By (70) and Proposition 14, we obtain for any $t \in [0, T_1]$ and $3 \leqslant \ell \leqslant k$,

$$\|(\varrho, u, \Theta)(t)\|_{H^\ell} \leqslant C_2 \|(\varrho_0, u_0, \Theta_0)\|_{H^\ell}, \tag{71}$$

which, together with (67), implies

$$\begin{gathered}(\varrho, u, \Theta)(T_1) \in H^3, \\ \|(\varrho, u, \Theta)(T_1)\|_{H^3} < \frac{\delta}{C_1}.\end{gathered} \tag{72}$$

Then, choosing $T_1 > 0$ as the new initial time instant, by Proposition 13 again, we obtain that the Cauchy problem (12)–(15) has a unique local solution

$$(\varrho, u, \Theta)(t) \in \mathscr{E}\left(T_1, 2T_1 ; H^3\right), \tag{73}$$

such that

$$\|(\varrho, u, \Theta)(t)\|_{H^3} \leqslant C_1 \|(\varrho, u, \Theta)(T_1)\|_{H^3} < \delta, \quad \forall t \in [T_1, 2T_1]. \tag{74}$$

From the above, we have proved that the Cauchy problem (12)–(15) has a unique local solution

$$(\varrho, u, \Theta)(t) \in \mathscr{E}\left(0, 2T_1 ; H^3\right), \tag{75}$$

such that

$$\|(\varrho, u, \Theta)(t)\|_{H^3} < \delta, \quad \forall t \in [0, 2T_1]. \tag{76}$$

By (76) and Proposition 14, we obtain for any $t \in [0, 2T_1]$ and $3 \leqslant \ell \leqslant k$,

$$\|(\varrho, u, \Theta)(t)\|_{H^\ell} \leqslant C_2 \|(\varrho_0, u_0, \Theta_0)\|_{H^\ell}, \tag{77}$$

which, together with (67) again, implies

$$\begin{gathered}(\varrho, u, \Theta)(2T_1) \in H^3, \\ \|(\varrho, u, \Theta)(2T_1)\|_{H^3} < \frac{\delta}{C_1}.\end{gathered} \tag{78}$$

By repeating the above procedures, we can extend the local solution to the global one only if $(\varrho_0, u_0, \Theta_0) \in H^k (k \geqslant 3)$ satisfying that $\|(\varrho_0, u_0, \Theta_0)\|_{H^3}$ is suitably small, as (67). So, we can choose $\delta_0 = \delta/(C_1 C_2)$ in Theorem 1. Hence, the proof of Theorem 1 is completed.

4. Decay Rates

In this section, we shall derive the decay rates (8) and (9) in Theorem 2. We will divide the proof into four parts.

4.1. Part 1: Decay Rates of the Solution Itself and Its Derivatives up to the $k-1$-Order. First, we show that the negative Sobolev or Besov norms of the solution $(\varrho, u, \Theta)(t)$ can be bounded by the initial data.

Lemma 15. *Under the assumptions of Theorem 1 and $g''(1) = 0$, if further $(\varrho_0, u_0, \Theta_0) \in \dot{H}^{-s}$ for some $s \in [0, 3/2)$ or $(\varrho_0, u_0, \Theta_0) \in \dot{B}^{-s}_{2,\infty}$ for some $s \in (0, 3/2]$, then for all $t \geqslant 0$,*

$$\|(\varrho, u, \Theta)(t)\|_{\dot{H}^{-s}} \leqslant C_0, \tag{79}$$

or

$$\|(\varrho, u, \Theta)(t)\|_{\dot{B}^{-s}_{2,\infty}} \leqslant C_0. \tag{80}$$

Proof. The detailed proof can be found in Appendix B. □

Then, we prove the following differential inequality with respect to time.

Lemma 16. *Let $k \geqslant 3$. Under the assumptions of Theorem 1, it holds that for all $t \geqslant 0$ and $0 \leqslant l \leqslant k-1$,*

$$\frac{d}{dt}\mathscr{E}_l^k(t) + \left\|\nabla^{l+1}\varrho(t)\right\|^2_{H^{k-l-1}} + \left\|\nabla^l (u, \Theta)(t)\right\|^2_{H^{k-l}} \leqslant 0, \tag{81}$$

where

$$\mathscr{E}_l^k(t) \sim \left\|\nabla^l (\varrho, u, \Theta)(t)\right\|^2_{H^{k-l}}. \tag{82}$$

Proof. It follows from Lemma 12 and Theorem 1. □

Next, we can use Lemmas 15 and 16 to prove the decay rates of the solution itself and its derivatives up to the $k-1$ -order. For $0 \leqslant l \leqslant k-1$, by Lemmas A.7 and A.8, we have

$$\left\|\nabla^{l+1} f\right\| \gtrsim \|f\|^{-1/(l+s)}_{\dot{H}^{-s}} \left\|\nabla^l f\right\|^{1+(1/(l+s))}, \tag{83}$$

$$\left\|\nabla^{l+1} f\right\| \gtrsim \|f\|_{\dot{B}_{2,\infty}^{-s}}^{-1/(l+s)}\left\|\nabla^{l} f\right\|^{1+(1/(l+s))}. \tag{84}$$

Combining (79) and (80) with (83) and (84), we have

$$\left\|\nabla^{l+1} \varrho\right\| \gtrsim C_0\left\|\nabla^{l} \varrho\right\|^{1+(1/(l+s))}. \tag{85}$$

This together with (7) infers for $0 \leqslant l \leqslant k-1$,

$$\left\|\nabla^{l+1} \varrho(t)\right\|_{H^{k-l-1}}^{2}+\left\|\nabla^{l}(u, \Theta)(t)\right\|_{H^{k-l}}^{2} \geqslant C_0\left(\left\|\nabla^{l}(\varrho, u, \Theta)\right\|_{H^{k-l}}^{2}\right)^{1+(1/(l+s))}. \tag{86}$$

From the differential inequality (81) of Lemma 16, we obtain for $0 \leqslant l \leqslant k-1$,

$$\frac{d}{dt} \mathscr{E}_l^k(t)+C_0\left(\mathscr{E}_l^k(t)\right)^{1+(1/(l+s))} \leqslant 0. \tag{87}$$

Solving the above inequality directly, we get for $0 \leqslant l \leqslant k-1$,

$$\mathscr{E}_l^k(t) \leqslant C_0(1+t)^{-(l+s)}. \tag{88}$$

By (82), we have for $0 \leqslant l \leqslant k-1$,

$$\left\|\nabla^{l}(\varrho, u, \Theta)(t)\right\|_{H^{k-l}}^{2} \leqslant C_0(1+t)^{-(l+s)}. \tag{89}$$

Note that the decay rate of the k-order derivatives is the same as one of the $k-1$-order.

4.2. Part 2: Higher Decay of u and Θ. We can further improve the decay rates of (u, Θ) as soon as we have the whole decay rate of (ϱ, u, Θ) by using the following processes.

By (13) and (14), we have

$$u_t+\frac{1}{\tau} u=-\nabla \Theta-\nabla \varrho-u \cdot \nabla u-\frac{\Theta-\varrho}{1+\varrho} \nabla \varrho, \tag{90}$$

$$\frac{1}{\gamma-1} \Theta_t+\frac{1}{\tau} \Theta=-\operatorname{div}\ u-\frac{1}{\gamma-1} u \cdot \nabla \Theta-\Theta \operatorname{div}\ u+\frac{1}{\tau} O(\varrho^3), \tag{91}$$

where we have used $g(1)=1$ and $g'(1)=g''(1)=0$ to obtain

$$g(\varrho+1)-1=O\left(\varrho^3\right). \tag{92}$$

Let $k \geqslant 3$ and $0 \leqslant l \leqslant k-2$. Applying ∇^l to (90) and (91), multiplying the resulting identities by $\nabla^l u$ and $\nabla^l \Theta$, respectively, and integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt}\left\|\nabla^{l} u\right\|^{2}+\frac{1}{\tau}\left\|\nabla^{l} u\right\|^{2} \\ &\quad=-\int \nabla^{l}\left(\nabla \Theta+\nabla \varrho+u \cdot \nabla u+\frac{\Theta-\varrho}{1+\varrho} \nabla \varrho\right) \cdot \nabla^{l} u, \\ &\frac{1}{\gamma-1} \frac{1}{2} \frac{d}{dt}\left\|\nabla^{l} \Theta\right\|^{2}+\frac{1}{\tau}\left\|\nabla^{l} \Theta\right\|^{2} \\ &\quad=-\int \nabla^{l}\left(\operatorname{div}\ u+\frac{1}{\gamma-1} u \cdot \nabla \Theta+\Theta \operatorname{div}\ u-\frac{1}{\tau} O\left(\varrho^{3}\right)\right) \cdot \nabla^{l} \Theta. \end{aligned} \tag{93}$$

By Hölder's, Sobolev's, and Cauchy's inequalities, Lemmas A.2 and A.3, and (89), we obtain

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^{l} u\right\|^{2}+C\left\|\nabla^{l} u\right\|^{2} \\ &\lesssim\left\|\left(\nabla^{l+1} \Theta, \nabla^{l+1} \varrho, \nabla^{l}(u \cdot \nabla u), \nabla^{l}\left(\frac{\Theta-\varrho}{1+\varrho} \nabla \varrho\right)\right)\right\|^{2} \\ &\lesssim\left\|\nabla^{l+1}(\varrho, \Theta)\right\|^{2}+\left\|\nabla^{l} u\right\|_{L^6}^{2}\|\nabla u\|_{L^3}^{2}+\|u\|_{L^\infty}^{2}\left\|\nabla^{l+1} u\right\|^{2} \\ &\quad+\left\|\nabla^{l}(\Theta-\varrho)\right\|_{L^6}^{2}\|\nabla \varrho\|_{L^3}^{2}+\|\Theta-\varrho\|_{L^\infty}^{2}\left\|\nabla^{l+1} \varrho\right\|^{2} \\ &\lesssim\left\|\nabla^{l+1}(\varrho, \Theta)\right\|^{2}+\|(\varrho, u, \Theta)\|_{H^2}^{2}\left\|\nabla^{l+1}(\varrho, u, \Theta)\right\|^{2} \\ &\leqslant C_0(1+t)^{-(l+1+s)}, \end{aligned} \tag{94}$$

$$\begin{aligned} &\frac{d}{dt}\left\|\nabla^{l} \Theta\right\|^{2}+C\left\|\nabla^{l} \Theta\right\|^{2} \\ &\lesssim\left\|\left(\nabla^{l+1} u, \nabla^{l}(u \cdot \nabla \Theta), \nabla^{l}(\Theta \operatorname{div}\ u), \nabla^{l} O\left(\varrho^{3}\right)\right)\right\|^{2} \\ &\lesssim\left\|\nabla^{l+1} u\right\|^{2}+\left\|\nabla^{l} u\right\|_{L^6}^{2}\|\nabla \Theta\|_{L^3}^{2}+\|u\|_{L^\infty}^{2}\left\|\nabla^{l+1} \Theta\right\|^{2} \\ &\quad+\left\|\nabla^{l} \Theta\right\|_{L^6}^{2}\|\nabla u\|_{L^3}^{2}+\|\Theta\|_{L^\infty}^{2}\left\|\nabla^{l+1} u\right\|^{2}+\left\|\nabla^{l} \varrho\right\|_{L^6}^{2}\|\varrho\|_{L^6}^{4} \\ &\lesssim\left\|\nabla^{l+1} u\right\|^{2}+\|(\varrho, u, \Theta)\|_{H^2}^{2}\left\|\nabla^{l+1}(\varrho, u, \Theta)\right\|^{2} \\ &\leqslant C_0(1+t)^{-(l+1+s)}. \end{aligned} \tag{95}$$

Applying Gronwall's inequality to (94) and (95), we obtain for $0 \leqslant l \leqslant k-2$,

$$\begin{aligned} \left\|\nabla^{l} u\right\|^{2} &\leqslant\left\|\nabla^{l} u_0\right\|^{2} e^{-Ct}+C_0 \int_0^t e^{-C(t-\varsigma)}(1+\varsigma)^{-(l+1+s)} \\ &\quad\cdot d\varsigma \leqslant C_0(1+t)^{-(l+1+s)}, \end{aligned} \tag{96}$$

$$\begin{aligned} \left\|\nabla^{l} \Theta\right\|^{2} &\leqslant\left\|\nabla^{l} \Theta_0\right\|^{2} e^{-Ct}+C_0 \int_0^t e^{-C(t-\varsigma)}(1+\varsigma)^{-(l+1+s)} \\ &\quad\cdot d\varsigma \leqslant C_0(1+t)^{-(l+1+s)}. \end{aligned} \tag{97}$$

4.3. Part 3: Decay Rates of the k-Order Derivatives of the Solution. We continue to derive the decay rate of the k

-order derivatives by using a time-weighted argument recently developed in [39].

Lemma 17. *Let $k \geqslant 3$ and s be given in Lemma 15. It holds that for $0 < \varepsilon_0 < 1$,*

$$(1+t)^{k+s}\left\|\nabla^k(\varrho,u,\Theta)\right\|^2 + C(1+t)^{-\varepsilon_0}\int_0^t (1+\tau)^{k+s+\varepsilon_0}\left\|\nabla^k(u,\Theta)\right\|^2 d\tau \leqslant C_0. \tag{98}$$

Proof. Let $k \geqslant 3$. As with the proof of Lemma 9, by using the known decay rates (89), (96), and (97), we easily obtain

$$\frac{d}{dt}\int \mathfrak{E}^k(t) + C\left\|\nabla^k(u,\Theta)\right\|^2 \leqslant C(1+t)^{-1}\left\|\nabla^k\varrho\right\|^2, \tag{99}$$

where

$$\begin{aligned}\mathfrak{E}^k(t) &= \frac{1+\Theta}{1+\varrho}\left|\nabla^k\varrho\right|^2 + (1+\varrho)\left|\nabla^{k-1}\operatorname{div} u\right|^2 + \left|\nabla^{k-1}\operatorname{curl} u\right|^2 \\ &\quad + \frac{1}{\gamma-1}\frac{1+\varrho}{1+\Theta}\left|\nabla^k\Theta\right|^2 \sim \left|\nabla^k(\varrho,u,\Theta)\right|^2.\end{aligned} \tag{100}$$

Let $0 < \varepsilon_0 < 1$. Multiplying (99) by $(1+t)^{k+s+\varepsilon_0}$ and integrating over $[0,t]$ in time, by (100), we have

$$\begin{aligned}&(1+t)^{k+s+\varepsilon_0}\left\|\nabla^k(\varrho,u,\Theta)\right\|^2 + C\int_0^t (1+\tau)^{k+s+\varepsilon_0}\left\|\nabla^k(u,\Theta)\right\|^2 \\ &\quad \cdot d\tau \leqslant \left\|\nabla^k(\varrho_0,u_0,\Theta_0)\right\|^2 + C\int_0^t (1+\tau)^{k+s+\varepsilon_0-1}\left\|\nabla^k(\varrho,u,\Theta)\right\|^2 d\tau.\end{aligned} \tag{101}$$

Next, we estimate the integral term on the right-hand side of (101). Letting $l = k-1$ in (81), we have

$$\frac{d}{dt}\mathscr{E}^k_{k-1}(t) + \left\|\nabla^k\varrho(t)\right\|^2 + \left\|\nabla^{k-1}(u,\Theta)(t)\right\|^2_{H^1} \leqslant 0, \tag{102}$$

where $\mathscr{E}^k_{k-1}(t) \sim \|\nabla^{k-1}(\varrho,u,\Theta)(t)\|^2_{H^1}$. Multiplying (102) by $(1+t)^{k+s+\varepsilon_0-1}$, by (88), we have

$$\begin{aligned}&\frac{d}{dt}\left[(1+t)^{k+s+\varepsilon_0-1}\mathscr{E}^k_{k-1}(t)\right] + (1+t)^{k+s+\varepsilon_0-1} \\ &\quad \cdot \left[\left\|\nabla^k\varrho(t)\right\|^2 + \left\|\nabla^{k-1}(u,\Theta)(t)\right\|^2_{H^1}\right] \\ &\leqslant (1+t)^{k+s+\varepsilon_0-2}\mathscr{E}^k_{k-1}(t) \leqslant C_0(1+t)^{-1+\varepsilon_0}.\end{aligned} \tag{103}$$

Integrating (103) over $[0,t]$ in time, we have

$$\begin{aligned}&(1+t)^{k+s+\varepsilon_0-1}\mathscr{E}^k_{k-1}(t) + \int_0^t (1+\tau)^{k+s+\varepsilon_0-1} \\ &\quad \cdot \left[\left\|\nabla^k\varrho\right\|^2 + \left\|\nabla^{k-1}(u,\Theta)\right\|^2_{H^1}\right] d\tau \leqslant \mathscr{E}^k_{k-1}(0) \\ &\quad + C_0\int_0^t (1+\tau)^{-1+\varepsilon_0} d\tau \leqslant C_0(1+t)^{\varepsilon_0},\end{aligned} \tag{104}$$

which implies

$$\int_0^t (1+\tau)^{k+s+\varepsilon_0-1}\left\|\nabla^k(\varrho,u,\Theta)\right\|^2 d\tau \leqslant C_0(1+t)^{\varepsilon_0}. \tag{105}$$

Plugging (105) into (101), we have

$$\begin{aligned}&(1+t)^{k+s+\varepsilon_0}\left\|\nabla^k(\varrho,u,\Theta)\right\|^2 + C\int_0^t (1+\tau)^{k+s+\varepsilon_0}\left\|\nabla^k(u,\Theta)\right\|^2 \\ &\quad \cdot d\tau \leqslant C_0(1+t)^{\varepsilon_0},\end{aligned} \tag{106}$$

which infers (98). □

By Lemma 17, we have

$$\left\|\nabla^k(\varrho,u,\Theta)(t)\right\|^2 \leqslant C_0(1+t)^{-(k+s)}. \tag{107}$$

4.4. Part 4: Conclusion. Repeating Part 2 (Section 4.2) with (107), we can obtain for $0 \leqslant l \leqslant k-1$,

$$\left\|\nabla^l(u,\Theta)(t)\right\|^2 \leqslant C_0(1+t)^{-(l+1+s)}. \tag{108}$$

Thus, the decay rates (8) and (9) of Theorem 2 hold from (89), (107), and (108). Hence, we complete the proof of Theorem 2.

Appendix

A. Tools

We will give some lemmas which are often used in the previous sections. We first recall the Gagliardo-Nirenberg-Sobolev inequality.

Lemma A.1. *Let $0 \leqslant m, \alpha \leqslant l$ and $2 \leqslant p \leqslant \infty$. Then, we have*

$$\|\nabla^\alpha f\|_{L^p} \lesssim \|\nabla^m f\|^{1-\vartheta}\left\|\nabla^l f\right\|^\vartheta, \tag{A.1}$$

where $0 \leqslant \vartheta \leqslant 1$ and α satisfy

$$\alpha + 3\left(\frac{1}{2} - \frac{1}{p}\right) = m(1-\vartheta) + l\vartheta. \tag{A.2}$$

Here, we require that $0 < \vartheta < 1$, $m \leqslant \alpha + 1$, and $l \geqslant \alpha + 2$, when $p = \infty$.

Proof. See [40] (Theorem, p. 125). □

We give the commutator and product estimates.

Lemma A.2. *Let m be a nonnegative integer. Define the commutator*

$$[\nabla^m, f]g := \nabla^m(fg) - f\nabla^m g. \tag{A.3}$$

Then, we have for $m \geq 1$,

$$\|[\nabla^m, f]g\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}} \left\|\nabla^{m-1} g\right\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{A.4}$$

and for $m \geq 0$,

$$\|\nabla^m(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|\nabla^m g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \tag{A.5}$$

where $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ and $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

Proof. Refer to [41] (Lemma 3.1) or [42] (Lemma A.4). □

The following lemma gives the convenient L^p estimates for well-prepared functions.

Lemma A.3. *Assume that $\|\varrho\|_{L^\infty} \leq 1$ and $\|\Theta\|_{L^\infty} \leq 1$. Let $f(\varrho, \Theta)$ be a smooth function of ϱ and Θ with bounded derivatives of any order; then, for any integer $k \geq 1$ and $2 \leq p \leq \infty$,*

$$\left\|\nabla^k(f(\varrho, \Theta))\right\|_{L^p} \lesssim \left\|\nabla^k \varrho\right\|_{L^p} + \left\|\nabla^k \Theta\right\|_{L^p}. \tag{A.6}$$

Proof. See [43] (Lemma A.2). □

As a byproduct of Lemma A.3, we immediately have the following.

Corollary A.4. *Assume that $\|\varrho\|_{L^\infty} \leq 1$. Let $f(\varrho)$ be a smooth function of ϱ with bounded derivatives of any order; then, for any integer $k \geq 1$ and $2 \leq p \leq \infty$,*

$$\left\|\nabla^k(f(\varrho))\right\|_{L^p} \lesssim \left\|\nabla^k \varrho\right\|_{L^p}. \tag{A.7}$$

Finally, we list some useful estimates or interpolation inequalities involving the negative Sobolev or Besov spaces.

Lemma A.5. *Let $1 < p \leq 2$ and $1/2 + s/3 = 1/p$. Then, $0 \leq s < 3/2$ and*

$$\|f\|_{\dot{H}^{-s}} \lesssim \|f\|_{L^p}. \tag{A.8}$$

Proof. It follows from the Hardy-Littlewood-Sobolev theorem (cf. [44] (Theorem 1, p. 119)). □

Lemma A.6. *Let $1 \leq p < 2$ and $1/2 + s/3 = 1/p$. Then, $0 < s \leq 3/2$ and*

$$\|f\|_{\dot{B}^{-s}_{2,\infty}} \lesssim \|f\|_{L^p}. \tag{A.9}$$

Proof. See [45] (Lemma 4.1). □

Lemma A.7. *Let $s \geq 0$ and $l \geq 0$. Then,*

$$\left\|\nabla^l f\right\| \lesssim \left\|\nabla^{l+1} f\right\|^{1-\vartheta} \|f\|^{\vartheta}_{\dot{H}^{-s}}, \quad \vartheta = \frac{1}{l+s+1}. \tag{A.10}$$

Proof. See [46] (Lemma A.4). □

Lemma A.8. *Let $s > 0$ and $l \geq 0$. Then,*

$$\left\|\nabla^l f\right\| \lesssim \left\|\nabla^{l+1} f\right\|^{1-\vartheta} \|f\|^{\vartheta}_{\dot{B}^{-s}_{2,\infty}}, \quad \vartheta = \frac{1}{l+s+1}. \tag{A.11}$$

Proof. We refer to [45] (Lemma 4.2) by noting that $\dot{B}^{-s}_{2,p} \subset \dot{B}^{-s}_{2,q}$ for $p \leq q$. □

B. Proof of Lemma 15

Here, we will prove Lemma 15. For this purpose, we first derive the negative Sobolev and Besov estimates of (ϱ, u, Θ) in the following lemmas.

Lemma B.1. *Suppose that $g(\rho)$ is a smooth function of ρ satisfying $g(1) = 1$ and $g'(1) = g''(1) = 0$. For $s \in (0, 1/2]$, we have*

$$\frac{d}{dt}\|(\varrho, u, \Theta)\|^2_{\dot{H}^{-s}} + C\|(u, \Theta)\|^2_{\dot{H}^{-s}} \lesssim \|\nabla(\varrho, u, \Theta)\|^2_{H^1} \|(\varrho, u, \Theta)\|_{\dot{H}^{-s}}, \tag{B.1}$$

and for $s \in (1/2, 3/2)$, we have

$$\begin{aligned} &\frac{d}{dt}\|(\varrho, u, \Theta)\|^2_{\dot{H}^{-s}} + C\|(u, \Theta)\|^2_{\dot{H}^{-s}} \\ &\lesssim \|(\varrho, u, \Theta)\|^{s-(1/2)} \|\nabla(\varrho, u, \Theta)\|^{(5/2)-s}_{H^1} \|(\varrho, u, \Theta)\|_{\dot{H}^{-s}}. \end{aligned} \tag{B.2}$$

Proof. Since $g(1) = 1$ and $g'(1) = g''(1) = 0$, equation (14) becomes

$$\frac{1}{\gamma - 1}\Theta_t + \frac{1}{\tau}\Theta = -\frac{1}{\gamma - 1} u \cdot \nabla\Theta - (1 + \Theta) \operatorname{div} u + \frac{1}{\tau} O(\varrho^3). \tag{B.3}$$

Applying Λ^{-s} to (12), (13), and (B.3), multiplying the resulting identities by $\Lambda^{-s}\varrho$, $\Lambda^{-s}u$, and $\Lambda^{-s}\Theta$, respectively, summing them up, and then integrating over $\mathbb{R}^3$ by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\left(\varrho, u, \frac{1}{\sqrt{\gamma-1}}\Theta\right)(t)\right\|_{\dot{H}^{-s}}^2+\frac{1}{\tau}\|(u,\Theta)(t)\|_{\dot{H}^{-s}}^2\\&=-\int\left[\Lambda^{-s}\operatorname{div}(\varrho u)\Lambda^{-s}\varrho+\Lambda^{-s}(u\cdot\nabla u)\cdot\Lambda^{-s}u\right.\\&\left.+\Lambda^{-s}\left(\frac{\Theta-\varrho}{1+\varrho}\nabla u\right)\cdot\Lambda^{-s}u\right]\\&-\int\left[\frac{1}{\gamma-1}\Lambda^{-s}(u\cdot\nabla\Theta)\Lambda^{-s}\Theta+\Lambda^{-s}(\operatorname{div}\ u\Theta)\Lambda^{-s}\Theta\right.\\&\left.-\frac{1}{\tau}\Lambda^{-s}\left(O\left(\varrho^3\right)\right)\Lambda^{-s}\Theta\right]\lesssim\|(\nabla\varrho\cdot u,\varrho\operatorname{div}\ u,u\\&\cdot\nabla u,\Theta\nabla u,\varrho\nabla u,u\cdot\nabla\Theta,\operatorname{div}\ u\Theta,\varrho^3)\|_{\dot{H}^{-s}}\|(\varrho,u,\Theta)\|_{\dot{H}^{-s}}.\end{aligned}\tag{B.4}$$

Then, we need to estimate the right-hand side of (B.4). If $s\in(0,1/2]$, then $(1/2)+(s/3)<1$ and $(3/s)\geqslant 6$. By Lemmas A.1 and A.5 and Hölder's and Young's inequalities, we have

$$\begin{aligned}&\|\nabla\varrho\cdot u\|_{\dot{H}^{-s}}\lesssim\|\nabla\varrho\cdot u\|_{L^{1/(1/2+s/3)}}\lesssim\|\nabla\varrho\|\|u\|_{L^{3/s}}\\&\lesssim\|\nabla\varrho\|\|\nabla u\|^{1/2+s}\left\|\nabla^2u\right\|^{1/2-s}\lesssim\|\nabla\varrho\|\|\nabla u\|_{H^1}\lesssim\|\nabla(\varrho,u)\|_{H^1}^2.\end{aligned}\tag{B.5}$$

Similarly, we obtain

$$\|(\varrho\operatorname{div}\ u,u\cdot\nabla u,\Theta\nabla u,\varrho\nabla u,u\cdot\nabla\Theta,\operatorname{div}\ u\Theta)\|_{\dot{H}^{-s}}\lesssim\|\nabla(\varrho,u,\Theta)\|_{H^1}^2,\tag{B.6}$$

$$\left\|\varrho^3\right\|_{\dot{H}^{-s}}\lesssim\left\|\varrho^2\right\|\|\varrho\|_{L^{s/3}}\lesssim\|\varrho\|_{L^3}\|\varrho\|_{L^6}\|\varrho\|_{L^{s/3}}\lesssim\|\nabla\varrho\|^2+\|\nabla\varrho\|_{H^1}^2,\tag{B.7}$$

where we have used $g''(1)=0$; otherwise, we have to encounter

$$\left\|\varrho^2\right\|_{\dot{H}^{-s}}\lesssim\|\varrho\|\|\varrho\|_{L^{s/3}}\lesssim\|\varrho\|\|\varrho\|_{L^{s/3}}\lesssim\|\varrho\|^2+\|\nabla\varrho\|_{H^1}^2,\tag{B.8}$$

and thus, this will make the later estimate (B.14) fail since ϱ is degenerately dissipative. Thus, plugging the estimates (B.5)–(B.7) into (B.4), we deduce (B.1).

Then, if $s\in(1/2,3/2)$, we will estimate the right-hand side of (B.4) in a different way. In this case, $(1/2)+(s/3)<1$ and $2<(3/s)<6$. Thus, by the different Sobolev interpolation, we easily obtain

$$\begin{aligned}&\left\|\left(\nabla\varrho\cdot u,\varrho\operatorname{div}\ u,u\cdot\nabla u,\Theta\nabla u,\varrho\nabla u,u\cdot\nabla\Theta,\operatorname{div}\ u\Theta,\varrho^3\right)\right\|_{\dot{H}^{-s}}\\&\lesssim\|\nabla(\varrho,u,\Theta)\|\|(\varrho,u,\Theta)\|^{s-(1/2)}\|\nabla(\varrho,u,\Theta)\|^{(3/2)-s}\\&\lesssim\|(\varrho,u,\Theta)\|^{s-(1/2)}\|\nabla(\varrho,u,\Theta)\|_{H^1}^{(5/2)-s}.\end{aligned}\tag{B.9}$$

Hence, plugging the estimate (B.9) into (B.4), we deduce (B.2). □

Lemma B.2. *Suppose that $g(\rho)$ is a smooth function of ρ satisfying $g(1)=1$ and $g'(1)=g''(1)=0$. For $s\in(0,1/2]$, we have*

$$\frac{d}{dt}\|(\varrho,u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}}^2+C\|(u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}}^2\lesssim\|\nabla(\varrho,u,\Theta)\|_{H^1}^2\|(\varrho,u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}},\tag{B.10}$$

and for $s\in(1/2,3/2]$, we have

$$\begin{aligned}&\frac{d}{dt}\|(\varrho,u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}}^2+C\|(u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}}^2\\&\lesssim\|(\varrho,u,\Theta)\|^{s-(1/2)}\|\nabla(\varrho,u,\Theta)\|_{H^1}^{(5/2)-s}\|(\varrho,u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}}.\end{aligned}\tag{B.11}$$

Proof. Applying $\dot{\Delta}_j$ to (12), (13), and (B.3), multiplying the resulting identities by $\dot{\Delta}_j\varrho$, $\dot{\Delta}_ju$, and $\dot{\Delta}_j\Theta$, respectively, summing them up, and then integrating over $\mathbb{R}^3$ by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\dot{\Delta}_j\left(\varrho,u,\frac{1}{\sqrt{\gamma-1}}\Theta\right)(t)\right\|^2+\frac{1}{\tau}\left\|\dot{\Delta}_j(u,\Theta)(t)\right\|^2\\&=-\int\left[\dot{\Delta}_j\operatorname{div}(\rho u)\dot{\Delta}_j\varrho+\dot{\Delta}_j(u\cdot\nabla u)\cdot\dot{\Delta}_ju+\dot{\Delta}_j\left(\frac{\Theta-\varrho}{1+\varrho}\nabla u\right)\cdot\dot{\Delta}_ju\right]\\&-\int\left[\frac{1}{\gamma-1}\dot{\Delta}_j(u\cdot\nabla\Theta)\dot{\Delta}_j\Theta+\dot{\Delta}_j(\operatorname{div}\ u\Theta)\dot{\Delta}_j\Theta-\frac{1}{\tau}\dot{\Delta}_j\left(O\left(\varrho^3\right)\right)\dot{\Delta}_j\Theta\right].\end{aligned}\tag{B.12}$$

Further, multiplying the above identity by 2^{-2sj} and then taking the supremum over $j\in\mathbb{Z}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\left(\varrho,u,\frac{1}{\sqrt{\gamma-1}}\Theta\right)(t)\right\|_{\dot{B}_{2,\infty}^{-s}}^2+\frac{1}{\tau}\|(u,\Theta)(t)\|_{\dot{B}_{2,\infty}^{-s}}^2\\&\lesssim\sup_{j\in\mathbb{Z}}2^{-2sj}\left\{-\int\left[\dot{\Delta}_j\operatorname{div}(\varrho u)\dot{\Delta}_j\varrho+\dot{\Delta}_j(u\cdot\nabla u)\cdot\dot{\Delta}_ju\right.\right.\\&\left.\left.+\dot{\Delta}_j\left(\frac{\Theta-\varrho}{1+\varrho}\nabla u\right)\cdot\dot{\Delta}_ju\right]\right\}+\sup_{j\in\mathbb{Z}}2^{-2sj}\\&\cdot\left\{-\int\left[\frac{1}{\gamma-1}\dot{\Delta}_j(u\cdot\nabla\Theta)\dot{\Delta}_j\Theta+\dot{\Delta}_j(\operatorname{div}\ u\Theta)\dot{\Delta}_j\Theta\right.\right.\\&\left.\left.-\frac{1}{\tau}\dot{\Delta}_j\left(O\left(\varrho^3\right)\right)\dot{\Delta}_j\Theta\right]\right\}\\&\lesssim\left\|\left(\nabla\varrho\cdot u,\varrho\operatorname{div}\ u,u\cdot\nabla u,\Theta\nabla u,\varrho\nabla u,u\cdot\nabla\Theta,\operatorname{div}\ u\Theta,\varrho^3\right)\right\|\\&\cdot_{\dot{B}_{2,\infty}^{-s}}\|(\varrho,u,\Theta)\|_{\dot{B}_{2,\infty}^{-s}}.\end{aligned}\tag{B.13}$$

The rest of the parts are totally similar to Lemma B.1 by replacing Lemma A.5 with Lemma A.6, so we omit it.? □

Now, we prove (79) and (80) in Lemma 15.

First, we prove (79) by Lemma B.1. For $s \in (0, 1/2]$, integrating (B.1) in time, by (7), we have

$$\begin{aligned}\|(\varrho, u, \Theta)(t)\|_{\dot{H}^{-s}}^2 &\leq \|(\varrho_0, u_0, \Theta_0)\|_{\dot{H}^{-s}}^2 \\ &+ C\int_0^t \|\nabla(\varrho, u, \Theta)(\varsigma)\|_{H^1}^2 \|(\varrho, u, \Theta)(\varsigma)\|_{\dot{H}^{-s}} \\ &\cdot d\varsigma \leq C_0\left(1 + \sup_{0\leq\varsigma\leq t}\|(\varrho, u, \Theta)(\varsigma)\|_{\dot{H}^{-s}}\right).\end{aligned} \tag{B.14}$$

By Young's inequality, we have

$$\|(\varrho, u, \Theta)(t)\|_{\dot{H}^{-s}}^2 \leq C_0, \quad \text{for } s \in \left[0, \frac{1}{2}\right], \tag{B.15}$$

and thus, this verifies (89) for $s \in [0, 1/2]$.

Then, we prove (79) for $s \in (1/2, 3/2)$. We have $(\varrho_0, u_0, \Theta_0) \in \dot{H}^{-1/2}$ since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$. We have proved (89) for $s \in [0, 1/2]$. Thus, when $s = 1/2$, we have

$$\left\|\nabla^l(\varrho, u, \Theta)(t)\right\|_{H^{k-l}}^2 \leq C_0(1+t)^{-(l+(1/2))}, \quad \text{for } -\frac{1}{2} \leq l \leq k-1. \tag{B.16}$$

By (B.16), integrating (B.2) in time, we obtain

$$\begin{aligned}\|(\varrho, u, \Theta)(t)\|_{\dot{H}^{-s}}^2 &\leq \|(\varrho_0, u_0, \Theta_0)\|_{\dot{H}^{-s}}^2 \\ &+ C\int_0^t \|(\varrho, u, \Theta)(\varsigma)\|^{s-(1/2)} \|\nabla(\varrho, u, \Theta)(\varsigma)\|_{H^1}^{(5/2)-s} \\ &\times \|(\varrho, u, \Theta)(\varsigma)\|_{\dot{H}^{-s}} d\varsigma \leq C_0 + C_0\int_0^t \\ &\cdot (1+\varsigma)^{-((7/4)-(s/2))}\, d\varsigma \sup_{0\leq\varsigma\leq t}\|(\varrho, u, \Theta)(\varsigma)\|_{\dot{H}^{-s}} \\ &\leq C_0\left(1 + \sup_{0\leq\varsigma\leq t}\|(\varrho, u, \Theta)(\varsigma)\|_{\dot{H}^{-s}}\right).\end{aligned} \tag{B.17}$$

By Young's inequality, we get

$$\|(\varrho, u, \Theta)(t)\|_{\dot{H}^{-s}}^2 \leq C_0, \quad \text{for } s \in \left(\frac{1}{2}, \frac{3}{2}\right), \tag{B.18}$$

and thus, this verifies (89) for $s \in (1/2, 3/2)$.

Next, we prove (80) by Lemma B.2. Similar to (79), we easily prove

$$\|(\varrho, u, \Theta)(t)\|_{\dot{B}_{2,\infty}^{-s}}^2 \leq C_0, \quad \text{for } s \in \left(0, \frac{3}{2}\right), \tag{B.19}$$

and thus, this verifies (89) for $s \in (1/2, 3/2)$. It remains to prove the case $s = 3/2$. Note that

$$\dot{B}_{2,\infty}^{-s} \cap L^2 \subset \dot{B}_{2,\infty}^{-s'}, \quad \text{for any } s' \in [0, s]. \tag{B.20}$$

For $s = 3/2$, we also have $(\varrho_0, u_0, \Theta_0) \in \dot{B}_{2,\infty}^{-1}$ due to (B.20). We have proved (80) and (89) for $s \in (0, 3/2)$. Therefore, when $s = 1$, we have

$$\left\|\nabla^l(\varrho, u, \Theta)(t)\right\|_{H^{k-l}}^2 \leq C_0(1+t)^{-(l+1)}, \quad \text{for } l = 0, 1. \tag{B.21}$$

So, by (B.21), integrating (B.11) in time, we get

$$\begin{aligned}\|(\varrho, u, \Theta)(t)\|_{\dot{B}_{2,\infty}^{-3/2}}^2 &\leq \|(\varrho_0, u_0, \Theta_0)\|_{\dot{B}_{2,\infty}^{-3/2}}^2 \\ &+ C\int_0^t \|(\varrho, u, \Theta)(\varsigma)\|^{s-(1/2)} \|\nabla(\varrho, u, \Theta)(\varsigma)\|_{H^1}^{(5/2)-s} \\ &\times \|(\varrho, u, \Theta)(\varsigma)\|_{\dot{B}_{2,\infty}^{-3/2}}\, d\varsigma \leq C_0 \\ &+ C_0\int_0^t (1+\varsigma)^{-(3/2)}\, d\varsigma \sup_{0\leq\varsigma\leq t}\|(\varrho, u, \Theta)(\varsigma)\|_{\dot{B}_{2,\infty}^{-3/2}} \\ &\leq C_0\left(1 + \sup_{0\leq\varsigma\leq t}\|(\varrho, u, \Theta)(\varsigma)\|_{\dot{B}_{2,\infty}^{-3/2}}\right).\end{aligned} \tag{B.22}$$

Similarly, this gives (80) and thus verifies (89) under $s = 3/2$.

Hence, the proof of Lemma 15 is completed.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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