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On Application of Matlab on Efficient Portfolio Management for a Pension Plan in the Presence of Uneven Distributions of Accumulated Wealth

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Authors' contributions

This work was carried out by the two authors. Authors OECM and AEE developed and solved the model. Author OECM did the simulations and wrote the literatures of the work. The two authors approved the final manuscript.

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Abstract

In this paper, we solved the problem encountered by a pension plan member whose portfolio is made up of one risk free asset and three risky assets for the optimal investment plan with return clause and uneven distributions of the remaining accumulated wealth. Using mean variance utility function as our objective function, we formulate our problem as a continuous-time mean–variance stochastic optimal control problem. Next, we used the variational inequalities methods to transform our problem into Markovian time inconsistent stochastic control, to determine the optimal investment plan and the efficient frontier of the plan member. Using mat lab software, we obtain numerical simulations of the optimal investment plan with respect to time and compare our results with an existing result.

Keywords: DC pension plan; mat lab; optimal investment plan; simulations; variational inequalities; return clause; mean variance utility.

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1 Introduction

In finding the optimal investment plan, we made use of some tools from the area of computer science and mathematics, which are applicable in virtually all fields. There are two major types of pension system; they include the defined benefit (DB) pension system and the defined contribution (DC) pension system. The DC pension plan requires its members to deposit some fractions of their income into their retirement savings account (RSA) over a specific time frame before their retirement from active service. These funds are further invested with the aim of increasing the members' wealth with the sole aim of meeting the members' needs after retirement. Due to the stochastic nature of the financial market, there is need for the pension fund administrators (PFA) to develop a unique investment plan that will guide them in carrying out their investment for optimal productivity. This has led to the study of optimal investment plan by different authors.

Many authors such as [1-4] among others studied the optimal investment problems under different assumptions. Investments in DC pension plan has evolve in the recent years, for example the study of optimal investment plan with return clause have taken centre stage since it takes into consideration the mortality risk of the members and this has drawn strength from the work of [5] where they studied optimal investment strategy with return of premium clause; they considered investment in one risk free asset and a risky asset modelled by geometric motions. In 2014, [6] studied optimal time-consistent investment strategy for a DC pension with the return of premiums clauses and annuity contracts; similar to [5]; they considered investment in a risk free asset and a risky asset but in their work, the risky asset was modelled by Heston volatility model. [7], studied optimization problem with return of premium in a DC pension with multiple contributors; in their work, the stock market price was driven by constant elasticity of variance model (CEV) model. [8], studied equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under (CEV) model; they considered investments in treasury, stock and bond. [9], studied optimal investment plan for a pension plan when the returned contributions are with predetermined interest; they considered investment in a risk free and a risky asset and assume the risky asset is modelled by Heston volatility model. [10], investigated investment plan with return of premium clauses under inflation risk and volatility risk; they considered investment in a risk free asset, the inflation index bond and the stock whose price was modelled by Heston volatility [11], studied optimal portfolio strategies with four assets modelled by geometric Brownian motion when the return premium is with interest and the remaining accumulated wealth are equally distributed among the surviving members. From the available literatures and to the best of our knowledge there is no work on the optimal investment plan with return clause that considers investment in four assets such that the return contributions are with predetermined interest and the remaining accumulations are not evenly distributed among the surviving members.

2 Portfolio Composition

Consider a complete market that frictionless and continuously open over a given time interval $t \in [0, T]$, where T is defined as the time frame of the accumulation phase. Let $(\Omega, \mathfrak{F}, \mathbb{p})$ be a complete probability space where Ω is a real space and $\mathbb{p}$ a probability measure, $\mathfrak{F}$ is the filtration representing the information generated by the Brownian motions$\{\mathcal{W}_1(t), \mathcal{W}_2(t), \mathcal{W}_3(t), \mathcal{W}_4(t), \mathcal{W}_5(t), \mathcal{W}_6(t)\}$.

Let $\mathcal{N}_0(t)$, $\mathcal{N}_1(t)$, $\mathcal{N}(t)$ and $\mathcal{N}_3(t)$ represent the prices of the risk-free asset and the three risky assets, and their models are given as follows:

$$d\mathcal{N}_0(t) = \mathcal{R}\mathcal{N}_0(t)dt, \tag{2.1}$$

$$d\mathcal{N}_1(t) = (\mathcal{R} + \ell_1)\mathcal{N}_1(t)dt + z_1\mathcal{N}_1(t)d\mathcal{W}_1(t). \tag{2.2}$$

$$d\mathcal{N}_2(t) = (\mathcal{R} + \ell_2)\mathcal{N}_2(t)dt + \mathcal{N}_2(t)(m_1 d\mathcal{W}_2(t) + m_2 d\mathcal{W}_3(t)) \tag{2.3}$$

$$d\mathcal{N}_3(t) = (\mathcal{R} + \ell_3)\mathcal{N}_3(t)dt + \mathcal{N}_3(t)(\text{n}_1 d\mathcal{W}_4(t) + \text{n}_2 d\mathcal{W}_5(t) + \text{n}_3 d\mathcal{W}_6(t)) \tag{2.4}$$

See [12]

Where $\mathcal{R}$ is a constant representing the risk-free interest rate, $(\mathcal{R} + \ell_1)$, $(\mathcal{R} + \ell_2)$ and $(\mathcal{R} + \ell_3)$ represent the expected instantaneous rate of return of the three risky assets and the instantaneous volatility of the risky assets are given by $z_1, m_1, m_2, \text{n}_1, \text{n}_2, \text{n}_3$. suppose that k is the members monthly contributions at any given time, $\hbar_0$ the initial age of accumulation phase and $\hbar_0 + T$ is the end age, $\frac{i}{i}\ddot{\Upsilon}_{\hbar_0+t}$ is the mortality rate from time t to $t + \frac{i}{i}$, tk is the accumulated contributions at time t, $tk\frac{i}{i}\ddot{\Upsilon}_{\hbar_0+t}$ is the premium returned to members next of kin.

Let $Q(t)$ represent the accumulated wealth of the pension fund at time t and d_0, d_1, d_2, and d_3 represent the fraction of the wealth invested in the four assets such that $d_0 = 1 - d_1 - d_2 - d_3$ is proportion invested in the risk free asset.

If we consider the time interval $[t, t + \frac{1}{i}]$, the differential form associated with the fund size when the remaining wealth is not evenly distributed among the remaining members is given as:

$$Q\left(t + \frac{1}{i}\right) = \left(\begin{array}{c} Q(t)\left(d_0 \frac{\mathcal{N}_0\left(t+\frac{1}{i}\right)}{\mathcal{N}_0(t)} + d_1 \frac{\mathcal{N}_1\left(t+\frac{1}{i}\right)}{\mathcal{N}_1(t)} + d_2 \frac{\mathcal{N}_2\left(t+\frac{1}{i}\right)}{\mathcal{N}_2(t)} + d_3 \frac{\mathcal{N}_3\left(t+\frac{1}{i}\right)}{\mathcal{N}_3(t)}\right) \\ +\frac{1}{i}k - atk\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t} - d_0 Q(t)\frac{\mathcal{N}_0\left(t+\frac{1}{i}\right)}{\mathcal{N}_0(t)}\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t} \end{array}\right) \tag{2.5}$$

$$Q\left(t + \frac{1}{i}\right) = \left(\begin{array}{c} Q(t)\left(\begin{array}{c} 1 + \left(\frac{\mathcal{N}_0\left(t+\frac{1}{i}\right) - \mathcal{N}_0(t)}{\mathcal{N}_0(t)}\right)(1 - \frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t})(1 - d_1 - d_2 - d_3) \\ + \left(\frac{\mathcal{N}_1\left(t+\frac{1}{i}\right) - \mathcal{N}_1(t)}{S_1(t)}\right)d_1 + \left(\frac{\mathcal{N}_2\left(t+\frac{1}{i}\right) - \mathcal{N}_2(t)}{\mathcal{N}_2(t)}\right)d_2 \\ + \left(\frac{\mathcal{N}_3\left(t+\frac{1}{i}\right) - \mathcal{N}_3(t)}{\mathcal{N}_3(t)}\right)d_2 \\ -Q(t)\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t}(1 - d_1 - d_2 - d_3) \end{array}\right) \\ +\frac{1}{i}k - atk\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t} \end{array}\right) \tag{2.6}$$

The conditional death probability ${}_tq_x = 1 - {}_tp_x = 1 - e^{-\int_0^t \boldsymbol{v}(\hbar_0+t+s)ds}$, where $\boldsymbol{v}(t)$ is the force function of the mortality at time t, and for $i \to \infty$,

$$\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t} = 1 - \exp\{-\int_0^{\frac{1}{i}} \boldsymbol{v}(\hbar_0 + t + s)ds\} \approx \boldsymbol{v}(\hbar_0 + t)\frac{1}{i} + O\frac{1}{i}$$

$$\frac{\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t}}{1-\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t}} = \frac{1-\exp\{-\int_0^{\frac{1}{i}} \boldsymbol{v}(\hbar_0+t+s)ds\}}{\exp\{-\int_0^{\frac{1}{i}} \boldsymbol{v}(\hbar_0+t+s)ds\}} = \exp\{\int_0^{\frac{1}{i}} \boldsymbol{v}(\hbar_0 + t + s)ds\} - 1 \approx \boldsymbol{v}(\hbar_0 + t)\frac{1}{i} = O(\frac{1}{i})$$

$$i \to \infty, \frac{\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t}}{1-\frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t}} = \boldsymbol{v}(\hbar_0 + t)dt, \frac{1}{i}\ddot{\Upsilon}_{\hbar_0+t} = \boldsymbol{v}(\hbar_0 + t)dt, \ \frac{1}{i}k \to kdt,$$

$$\frac{\mathcal{N}_n(t+n) - \mathcal{N}_n(t)}{\mathcal{N}_n(t)} \to \frac{d\mathcal{N}_n(t)}{\mathcal{N}_n(t)}, \text{ for } n = 0,1,2,3 \tag{2.7}$$

Substituting (2.7) into (2.6) we have

$$\mathcal{Q}\left(t+\frac{1}{i}\right)=\begin{pmatrix} \mathcal{Q}(t)\begin{pmatrix} 1+(1-d_1-d_2-d_3)(1-\boldsymbol{v}(\hbar_0+t)dt)\frac{d\mathcal{N}_0(t)}{\mathcal{N}_0(t)} \\ +d_1\frac{d\mathcal{N}_1(t)}{\mathcal{N}_1(t)}+d_2\frac{d\mathcal{N}_2(t)}{\mathcal{N}_2(t)}+d_3\frac{d\mathcal{N}_3(t)}{\mathcal{N}_3(t)} \end{pmatrix} \\ +kdt-atk\boldsymbol{v}(\hbar_0+t)dt \\ -(1-d_1-d_2-d_3)\mathcal{Q}(t)\boldsymbol{v}(\hbar_0+t)dt \end{pmatrix} \tag{2.8}$$

$$d\mathcal{Q}(t)=\begin{cases} \begin{pmatrix} \mathcal{Q}(t)\begin{pmatrix} \mathcal{R}-\frac{1}{\hbar-\hbar_0-t}+d_1(\ell_1+\frac{1}{\hbar-\hbar_0-t})+d_2(\ell_2+\frac{1}{\hbar-\hbar_0-t}) \\ +d_3(\ell_3+\frac{1}{\hbar-\hbar_0-t}) \end{pmatrix} \\ +k\left(\frac{\hbar-\hbar_0-(a+1)t}{\hbar-\hbar_0-t}\right) \end{pmatrix}dt \\ +\mathcal{Q}(t)\begin{pmatrix} d_1z_1d\mathcal{W}_1(t)+d_2\left(m_1d\mathcal{W}_2(t)+m_2d\mathcal{W}_3(t)\right) \\ +d_3\left(\mathrm{n}_1d\mathcal{W}_4(t)+\mathrm{n}_2d\mathcal{W}_5(t)+\mathrm{n}_3d\mathcal{W}_6(t)\right) \end{pmatrix} \end{cases} \mathcal{Q}(0)=q_0 \tag{2.9}$$

Where $\boldsymbol{v}(t)$ is the force function and $\hbar$ is the maximal age of the life table and are related as follows

$$\boldsymbol{v}(t)=\frac{1}{\hbar+t} \quad 0\le t<\hbar \tag{2.10}$$

3 Mean Variance Utility, Optimal Investment Plan and Efficient Frontier

Based on the fact that surviving member interest is to increase the fund size at end of accumulation period, it is necessary for fund managers to formulate an optimal portfolio problem under the mean-variance condition as follows:

$$A(t,q)=\sup_d\left\{E_{t,q}\mathcal{Q}^d(T)-Var_{t,q}\mathcal{Q}^d(T)\right\} \tag{3.1}$$

Observe that (3.1) is similar to Markovian time inconsistent stochastic optimal control problem with value function $A(t,q)$.

$$\begin{cases} B(t,q,d)=E_{t,q}[\mathcal{Q}^d(T)]-\frac{\beta}{2}Var_{t,q}[\mathcal{Q}^d(T)] \\ B(t,q,d)\ =E_{t,q}[\mathcal{Q}^d(T)]-\frac{\beta}{2}\left(E_{t,q}[\mathcal{Q}^d(T)^2]-(E_{t,q}[\mathcal{Q}^d(T)])^2\right) \\ A(t,q)\ =\sup_d B(t,q,d) \end{cases} \tag{3.2}$$

Following the procedure in [13], the optimal investment strategy d^* satisfies:

$$A(t,q)=\sup_d B(t,q,d) \tag{3.3}$$

β is a constant representing risk aversion coefficient of the members.

Let $u^d(t,q)=E_{t,q}[\mathcal{Q}^d(T)]$, $v^d(t,q)=E_{t,q}[\mathcal{Q}^d(T)^2]$ then

$$A(t,q)\quad=\sup_d x\left(t,q,u^d(t,q),v^d(t,q)\right)$$

Where,

$$x(t,q,u,v)=u-\frac{\beta}{2}(v-u^2)$$

Applying game theoretic method in [13], we establish the extended Hamilton Jacobi Bellman equation which is a system of non linear PDE.

Theorem 3.1 (verification theorem) [5,13]: If there exist three real functions $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ $[0,T] \times R \rightarrow R$ satisfying the following extended Hamilton Jacobi Bellman equations:

$$\begin{cases} \sup_{d} \left\{ \begin{array}{c} \mathcal{X}_t - x_t + (\mathcal{X}_q - x_q) \left[q \begin{pmatrix} \mathcal{R} - \frac{1}{\hbar - \hbar_0 - t} + d_1(\ell_1 + \frac{1}{\hbar - \hbar_0 - t}) \\ + d_2(\ell_2 + \frac{1}{\hbar - \hbar_0 - t}) \\ + d_3(\ell_3 + \frac{1}{\hbar - \hbar_0 - t}) \end{pmatrix} + k\left(\frac{\hbar - \hbar_0 - (a+1)t}{\hbar - \hbar_0 - t}\right) \right] \\ + \frac{1}{2}(\mathcal{X}_{qq} - \mathcal{M}_{qq})(d_1^2 z_1^2 + d_2^2(m_1^2 + m_2^2) + d_3^2(n_2^2 + n_2^2 + n_2^2)) \end{array} \right\} = 0 \\ \mathcal{X}(T, q) = x(t, q, q, q^2) \end{cases} \quad (3.4)$$

Where,

$$\mathcal{M}_{qq} = x_{qq} + 2x_{qu}u_l + 2x_{qv}v_q + x_{uu}u_q^2 + 2x_{uv}u_q v_q + x_{vv}v_q^2 = \beta u_q^2$$

$$\begin{cases} \left\{ \begin{array}{c} \mathcal{Y}_t + \mathcal{Y}_q \left[q \begin{pmatrix} \mathcal{R} - \frac{1}{\hbar - \hbar_0 - t} + d_1(\ell_1 + \frac{1}{\hbar - \hbar_0 - t}) \\ + d_2(\ell_2 + \frac{1}{\hbar - \hbar_0 - t}) \\ + d_3(\ell_3 + \frac{1}{\hbar - \hbar_0 - t}) \end{pmatrix} + k\left(\frac{\hbar - \hbar_0 - (a+1)t}{\hbar - \hbar_0 - t}\right) \right] \\ + \frac{1}{2}\mathcal{Y}_{qq}(d_1^2 z_1^2 + d_2^2(m_1^2 + m_2^2) + d_3^2(n_2^2 + n_2^2 + n_2^2)) \end{array} \right\} = 0 \\ \mathcal{Y}(T, q) = q \end{cases} \quad (3.5)$$

$$\begin{cases} \left\{ \begin{array}{c} \mathcal{Z}_t + \mathcal{Z}_q \left[q \begin{pmatrix} \mathcal{R} - \frac{1}{\hbar - \hbar_0 - t} + d_1(\ell_1 + \frac{1}{\hbar - \hbar_0 - t}) \\ + d_2(\ell_2 + \frac{1}{\hbar - \hbar_0 - t}) \\ + d_3(\ell_3 + \frac{1}{\hbar - \hbar_0 - t}) \end{pmatrix} + k\left(\frac{\hbar - \hbar_0 - (a+1)t}{\hbar - \hbar_0 - t}\right) \right] \\ + \frac{1}{2}\mathcal{Z}_{qq}(d_1^2 z_1^2 + d_2^2(m_1^2 + m_2^2) + d_3^2(n_2^2 + n_2^2 + n_2^2)) \end{array} \right\} = 0 \\ \mathcal{Z}(T, q) = q^2 \end{cases} \quad (3.6)$$

Then $A(t,q) = \mathcal{X}(t,q), u^{d^*} = \mathcal{Y}(t,q), v^{d^*} = \mathcal{Z}(t,q)$ for the optimal investment plan d^*

Proof:

The details of the proof can be found in [14,15,16].

Next we proceed to solve the extended HJB equations for the optimal investment plan and the efficient frontier of the pension member.

Recall that $x(t,q,u,v) = u - \frac{\beta}{2}(v - u^2)$

Differentiating $x(t,q,u,v)$ with respect to the variables t, q, u, v, we have

$$x_t = x_q = x_{qq} = x_{qu} = x_{qv} = x_{uv} = x_{vv} = 0, x_u = 1 + \beta u, x_{uu} = \beta, x_v = -\frac{\beta}{2} \quad (3.7)$$

Substituting (3.7) into (3.4) and differentiating (3.4) with respect to d_1, d_2 and d_3 and solving for d_1, d_2 and d_3, we have

$$d_1{}^* = -\left[\frac{(\ell_1+\frac{1}{\hbar-\hbar_0-t})\mathcal{X}_q}{(\mathcal{X}_{qq}-\beta \mathcal{Y}_q^2)q z_1^2}\right] \tag{3.8}$$

$$d_2{}^* = -\left[\frac{(\ell_2+\frac{1}{\hbar-\hbar_0-t})\mathcal{X}_q}{(\mathcal{X}_{qq}-\beta \mathcal{Y}_q^2)q(m_1^2+m_2^2)}\right] \tag{3.9}$$

$$d_3{}^* = -\left[\frac{(\ell_3+\frac{1}{\hbar-\hbar_0-t})\mathcal{X}_q}{(\mathcal{X}_{qq}-\beta \mathcal{Y}_q^2)q(n_2^2+n_2^2+n_2^2)}\right] \tag{3.10}$$

Substituting (3.8), (3.9) and (3,10) into (3.4) and (3.5) we have

$$\mathcal{X}_t + \mathcal{X}_q\left[\left(\mathcal{R}-\frac{1}{\hbar-\hbar_0-t}\right)q + k\left(\frac{\hbar-\hbar_0-(a+1)t}{\hbar-\hbar_0-t}\right)\right] - \frac{\mathcal{X}_q^2}{2(\mathcal{X}_{qq}-\beta\mathcal{Y}_q^2)}\left(\frac{(\ell_1+\frac{1}{\hbar-\hbar_0-t})^2}{z_1^2}+\frac{(\ell_2+\frac{1}{\hbar-\hbar_0-t})^2}{(m_1^2+m_2^2)}+\frac{(\ell_3+\frac{1}{\hbar-\hbar_0-t})^2}{(n_2^2+n_2^2+n_2^2)}\right)=0 \tag{3.11}$$

$$\mathcal{Y}_t + \mathcal{Y}_q\left[\left(\mathcal{R}-\frac{1}{\hbar-\hbar_0-t}\right)q + k\left(\frac{\hbar-\hbar_0-(a+1)t}{\hbar-\hbar_0-t}\right)\right] - \frac{\mathcal{X}_q\mathcal{Y}_q}{(\mathcal{X}_{qq}-\beta\mathcal{Y}_q^2)}\left(\frac{(\ell_1+\frac{1}{\hbar-\hbar_0-t})^2}{z_1^2}+\frac{(\ell_2+\frac{1}{\hbar-\hbar_0-t})^2}{(m_1^2+m_2^2)}+\frac{(\ell_3+\frac{1}{\hbar-\hbar_0-t})^2}{(n_2^2+n_2^2+n_2^2)}\right)$$
$$+\frac{\mathcal{Y}_{qq}}{2}\left[\frac{\mathcal{X}_q^2}{(\mathcal{X}_{qq}-\beta\mathcal{Y}_q^2)}\left(\frac{(\ell_1+\frac{1}{\hbar-\hbar_0-t})^2}{z_1^2}+\frac{(\ell_2+\frac{1}{\hbar-\hbar_0-t})^2}{(m_1^2+m_2^2)}+\frac{(\ell_3+\frac{1}{\hbar-\hbar_0-t})^2}{(n_2^2+n_2^2+n_2^2)}\right)\right]=0 \tag{3.12}$$

If we assume solutions for $\mathcal{X}(t,q)$ and $\mathcal{Y}(t,q)$ as follows:

$$\begin{cases} \mathcal{X}(t,q) = C(t)q + \frac{D(t)}{\beta} \quad C(T)=1,\ D(T)=0 \\ \mathcal{Y}(t,q) = E(t)q + \frac{F(t)}{\beta} \quad E(T)=1,\ F(T)=0 \\ \mathcal{X}_t = qC_t(t) + \frac{D_t(t)}{\beta}, \mathcal{X}_q = C(t), \mathcal{X}_{qq}=0,\ \mathcal{Y}_t = qE_t(t)+\frac{F_t(t)}{\beta}, \mathcal{Y}_q = E(t), \mathcal{Y}_{qq}=0 \end{cases} \tag{3.13}$$

Substituting (3.13) into (3.11) and (3.12), we have

$$\begin{cases} C_t(t) + \left(\mathcal{R}-\frac{1}{\hbar-\hbar_0-t}\right)C(t)=0 \\ D_t(t) + C(t)k\beta\left(\frac{\hbar-\hbar_0-(a+1)t}{\hbar-\hbar_0-t}\right)+\frac{C^2(t)}{2E^2(t)}\left(\frac{(\ell_1+\frac{1}{\hbar-\hbar_0-t})^2}{z_1^2}+\frac{(\ell_2+\frac{1}{\hbar-\hbar_0-t})^2}{(m_1^2+m_2^2)}+\frac{(\ell_3+\frac{1}{\hbar-\hbar_0-t})^2}{(n_2^2+n_2^2+n_2^2)}\right)=0 \end{cases} \tag{3.14}$$

$$\begin{cases} E_t(t) + \left(\mathcal{R}-\frac{1}{\hbar-\hbar_0-t}\right)E(t)=0 \\ F_t(t) + E(t)k\beta\left(\frac{\hbar-\hbar_0-(a+1)t}{\hbar-\hbar_0-t}\right)+\frac{C(t)}{E(t)}\left(\frac{(\ell_1+\frac{1}{\hbar-\hbar_0-t})^2}{z_1^2}+\frac{(\ell_2+\frac{1}{\hbar-\hbar_0-t})^2}{(m_1^2+m_2^2)}+\frac{(\ell_3+\frac{1}{\hbar-\hbar_0-t})^2}{(n_2^2+n_2^2+n_2^2)}\right)=0 \end{cases} \tag{3.15}$$

Solving (3.14) and (3.15), we have

$$C(t) = \left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \tag{3.16}$$

$$E(t) = \left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \tag{3.17}$$

$$D(t) = \left(\begin{array}{c} k\beta(\hbar-\hbar_0-T)\int_t^T\left(\frac{\hbar-\hbar_0-(a+1)\tau}{(\hbar-\hbar_0-\tau)^2}\right)e^{\mathcal{R}(T-t)}d\tau \\ +\frac{1}{2}\left[\begin{array}{c} \left(\frac{\ell_1{}^2}{z_1^2}+\frac{\ell_2{}^2}{(m_1^2+m_2^2)}+\frac{\ell_3{}^2}{(n_2^2+n_2^2+n_2^2)}\right)(T-t)+ \\ \left(\frac{\ell_1}{z_1^2}+\frac{\ell_2}{(m_1^2+m_2^2)}+\frac{\ell_3}{(n_2^2+n_2^2+n_2^2)}\right)ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2+ \\ \left(\frac{1}{z_1^2}+\frac{1}{(m_1^2+m_2^2)}+\frac{1}{(n_2^2+n_2^2+n_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right) \end{array}\right] \end{array}\right) \tag{3.18}$$

$$F(t)=\begin{pmatrix} k\beta(\hbar-\hbar_0-T)\int_t^T\left(\frac{\hbar-\hbar_0-(a+1)\tau}{(\hbar-\hbar_0-\tau)^2}\right)e^{\mathcal{R}(T-t)}d\tau \\ +\begin{bmatrix} \left(\frac{\ell_1{}^2}{z_1^2}+\frac{\ell_2{}^2}{(m_1^2+m_2^2)}+\frac{\ell_3{}^2}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)(T-t)+ \\ \left(\frac{\ell_1}{z_1^2}+\frac{\ell_2}{(m_1^2+m_2^2)}+\frac{\ell_3}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2+ \\ \left(\frac{1}{z_1^2}+\frac{1}{(m_1^2+m_2^2)}+\frac{1}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{bmatrix}\end{pmatrix} \quad (3.19)$$

$$\mathcal{X}(t,q)=\begin{pmatrix} q\left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \\ +k(\hbar-\hbar_0-T)\int_t^T\left(\frac{\hbar-\hbar_0-(a+1)\tau}{(\hbar-\hbar_0-\tau)^2}\right)e^{\mathcal{R}(T-t)}d\tau \\ +\frac{1}{2\beta}\begin{bmatrix} \left(\frac{\ell_1{}^2}{z_1^2}+\frac{\ell_2{}^2}{(m_1^2+m_2^2)}+\frac{\ell_3{}^2}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)(T-t)+ \\ \left(\frac{\ell_1}{z_1^2}+\frac{\ell_2}{(m_1^2+m_2^2)}+\frac{\ell_3}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2+ \\ \left(\frac{1}{z_1^2}+\frac{1}{(m_1^2+m_2^2)}+\frac{1}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{bmatrix}\end{pmatrix} \quad (3.20)$$

$$\mathcal{Y}(t,q)=\begin{pmatrix} q\left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \\ +k(\hbar-\hbar_0-T)\int_t^T\left(\frac{\hbar-\hbar_0-(a+1)\tau}{(\hbar-\hbar_0-\tau)^2}\right)e^{\mathcal{R}(T-t)}d\tau \\ +\frac{1}{\beta}\begin{bmatrix} \left(\frac{\ell_1{}^2}{z_1^2}+\frac{\ell_2{}^2}{(m_1^2+m_2^2)}+\frac{\ell_3{}^2}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)(T-t)+ \\ \left(\frac{\ell_1}{z_1^2}+\frac{\ell_2}{(m_1^2+m_2^2)}+\frac{\ell_3}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2+ \\ \left(\frac{1}{z_1^2}+\frac{1}{(m_1^2+m_2^2)}+\frac{1}{(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{bmatrix}\end{pmatrix} \quad (3.21)$$

From (3.13), we have

$$\mathcal{X}_q=\left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)},\mathcal{X}_{qq}=0,\mathcal{Y}_q=\left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \quad (3.22)$$

Substituting (3.22) into (3.8), (3.9) and (3.10), we have

$$d_1{}^*=\left(\frac{\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\ell_1+\frac{1}{\hbar-\hbar_0-T}}{\beta q z_1^2}\right)e^{\mathcal{R}(t-T)} \quad (3.23)$$

$$d_2{}^*=\left(\frac{\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\ell_2+\frac{1}{\hbar-\hbar_0-T}}{\beta q(m_1^2+m_2^2)}\right)e^{\mathcal{R}(t-T)} \quad (3.24)$$

$$d_3{}^*=\left(\frac{\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\ell_3+\frac{1}{\hbar-\hbar_0-T}}{\beta q(\text{n}_2^2+\text{n}_2^2+\text{n}_2^2)}\right)e^{\mathcal{R}(t-T)} \quad (3.25)$$

$$d_0{}^*=1-d_1{}^*-d_2{}^*-d_3{}^* \quad (3.26)$$

Also we proceed to solve for the efficient frontier of the pension fund which gives the relationship between the expectation and variance.

Recall that

$$Var_{t,q}[Q^{d^*}(T)] = E_{t,q}[Q^{d^*}(T)^2] - (E_{t,q}[Q^{d^*}(T)])^2$$

$$Var_{t,q}[Q^{d^*}(T)] = \frac{2}{\beta}(\mathcal{Y}(t,q) - \mathcal{X}(t,q)) \tag{3.27}$$

Substituting (3.20) and (3.21) into (3.27), we have

$$Var_{t,q}[Q^{d^*}(T)] = \frac{1}{\beta^2}\left[\begin{array}{c}\left(\frac{{\ell_1}^2}{z_1^2} + \frac{{\ell_2}^2}{(m_1^2+m_2^2)} + \frac{{\ell_3}^2}{(n_2^2+n_2^2+n_2^2)}\right)(T-t) + \\ \left(\frac{\ell_1}{z_1^2} + \frac{\ell_2}{(m_1^2+m_2^2)} + \frac{\ell_3}{(n_2^2+n_2^2+n_2^2)}\right) ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2 + \\ \left(\frac{1}{z_1^2} + \frac{1}{(m_1^2+m_2^2)} + \frac{1}{(n_2^2+n_2^2+n_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{array}\right] \tag{3.28}$$

$$\frac{1}{\beta} = \sqrt{\frac{Var_{t,q}[Q^{d^*}(T)]}{\left[\begin{array}{c}\left(\frac{{\ell_1}^2}{z_1^2} + \frac{{\ell_2}^2}{(m_1^2+m_2^2)} + \frac{{\ell_3}^2}{(n_2^2+n_2^2+n_2^2)}\right)(T-t) + \\ \left(\frac{\ell_1}{z_1^2} + \frac{\ell_2}{(m_1^2+m_2^2)} + \frac{\ell_3}{(n_2^2+n_2^2+n_2^2)}\right) ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2 + \\ \left(\frac{1}{z_1^2} + \frac{1}{(m_1^2+m_2^2)} + \frac{1}{(n_2^2+n_2^2+n_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{array}\right]}} \tag{3.29}$$

$$E_{t,q}[Q^{d^*}(T)] = \mathcal{Y}(t,q) \tag{3.30}$$

Substituting (3.21) into (3.30), we have

$$E_{t,q}[Q^{d^*}(T)] = \left(\begin{array}{c} q\left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \\ +k(\hbar-\hbar_0-T)\int_t^T\left(\frac{\hbar-\hbar_0-(a+1)\tau}{(\hbar-\hbar_0-\tau)^2}\right)e^{\mathcal{R}(T-t)}d\tau \\ +\frac{1}{\beta}\left[\begin{array}{c}\left(\frac{{\ell_1}^2}{z_1^2} + \frac{{\ell_2}^2}{(m_1^2+m_2^2)} + \frac{{\ell_3}^2}{(n_2^2+n_2^2+n_2^2)}\right)(T-t) + \\ \left(\frac{\ell_1}{z_1^2} + \frac{\ell_2}{(m_1^2+m_2^2)} + \frac{\ell_3}{(n_2^2+n_2^2+n_2^2)}\right) ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2 + \\ \left(\frac{1}{z_1^2} + \frac{1}{(m_1^2+m_2^2)} + \frac{1}{(n_2^2+n_2^2+n_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{array}\right]\end{array}\right) \tag{3.31}$$

Substituting (3.29) into (3.31), we have:

$$E_{t,q}[Q^{d^*}(T)] = \left(\begin{array}{c} q\left(\frac{\hbar-\hbar_0-T}{\hbar-\hbar_0-t}\right)e^{\mathcal{R}(T-t)} \\ +k(\hbar-\hbar_0-T)\int_t^T\left(\frac{\hbar-\hbar_0-(a+1)\tau}{(\hbar-\hbar_0-\tau)^2}\right)e^{\mathcal{R}(T-t)}d\tau \\ +\sqrt{\left[\begin{array}{c}\left(\frac{{\ell_1}^2}{z_1^2} + \frac{{\ell_2}^2}{(m_1^2+m_2^2)} + \frac{{\ell_3}^2}{(n_2^2+n_2^2+n_2^2)}\right)(T-t) + \\ \left(\frac{\ell_1}{z_1^2} + \frac{\ell_2}{(m_1^2+m_2^2)} + \frac{\ell_3}{(n_2^2+n_2^2+n_2^2)}\right) ln\left(\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}\right)^2 + \\ \left(\frac{1}{z_1^2} + \frac{1}{(m_1^2+m_2^2)} + \frac{1}{(n_2^2+n_2^2+n_2^2)}\right)\left(\frac{T-t}{(\hbar-\hbar_0-t)(\hbar-\hbar_0-T)}\right)\end{array}\right]Var_{t,q}[Q^{d^*}(T)]}\end{array}\right) \tag{3.32}$$

Remark 1

The optimal investment plan with even distribution of accumulated wealth is given as

$${J_0}^* = 1 - {J_1}^* - {J_2}^* - {J_3}^*$$

$${J_1}^* = \left(\frac{\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}}{\beta q z_1^2}\right)\ell_1 e^{\mathcal{R}(t-T)}$$

$$J_2{}^* = d_2{}^* = \left(\frac{\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}}{\beta q(m_1^2+m_2^2)}\right)\ell_2 e^{\mathcal{R}(t-T)}$$

$$J_3{}^* = d_3{}^* = \left(\frac{\frac{\hbar-\hbar_0-t}{\hbar-\hbar_0-T}}{\beta q(n_2^2+n_2^2+n_2^2)}\right)\ell_3 e^{\mathcal{R}(t-T)}$$

See [11]

4 Numerical Simulations

Here, we present numerical simulations of the optimal investment plan with respect to time using Matlab programming language making use of the following parameters. $\hbar$ = 100; $\hbar_0$ = 20;β = 0.05; $\mathcal{R}$ = 0.02; k = 1; ℓ_1 = 0.035;ℓ_2 = 0.045; ℓ_3 = 0.055;$z_1 = 0.85$; m_1 = 1; m_2= 0.60; n_1= 1.15, n_2= 0.75; n_3= 0.40; $q = Q(t)$; $q_0 = 1$; $T = 40$; $t = 0.5{:}\,20$.

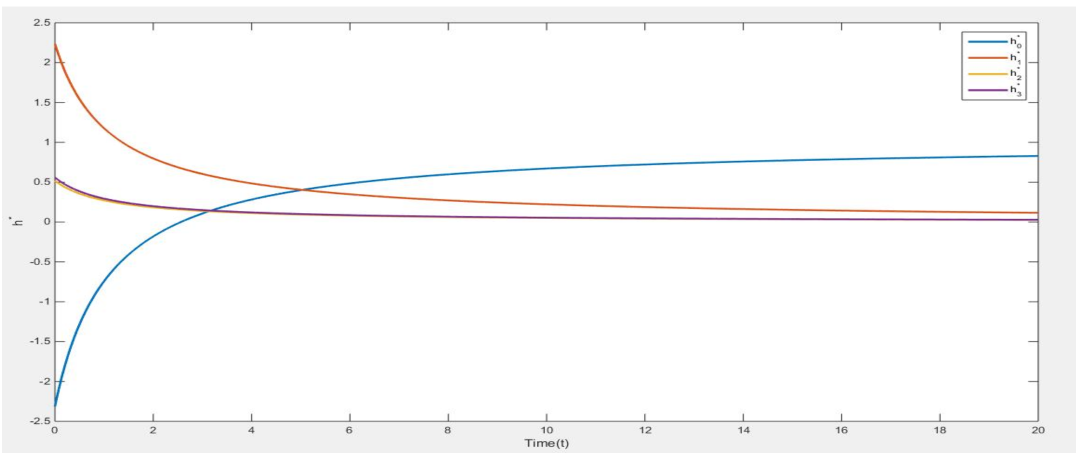

Fig. 1. Optimal investment plan with uneven distribution of remaining accumulated wealth when $q = Q(t)$

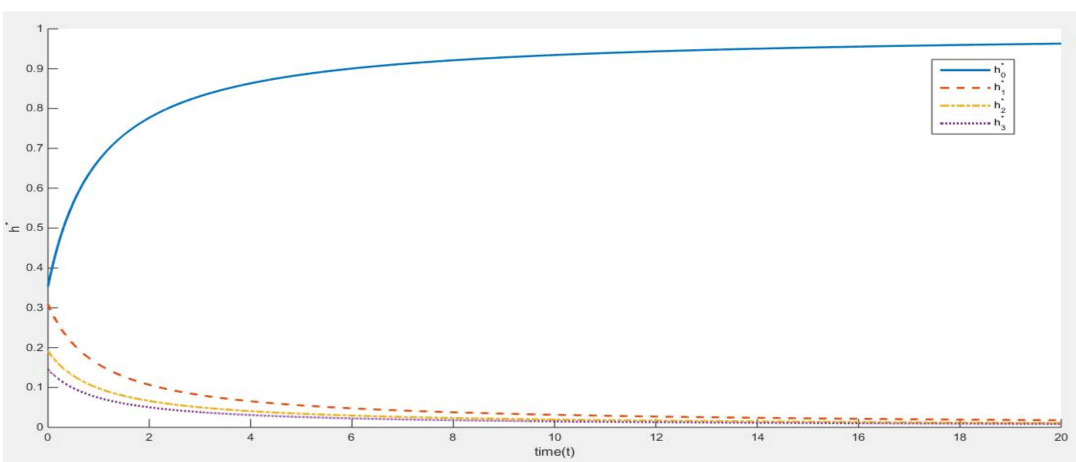

Fig. 2. Optimal investment plan with equal distribution of remaining accumulated wealth when $q = Q(t)$

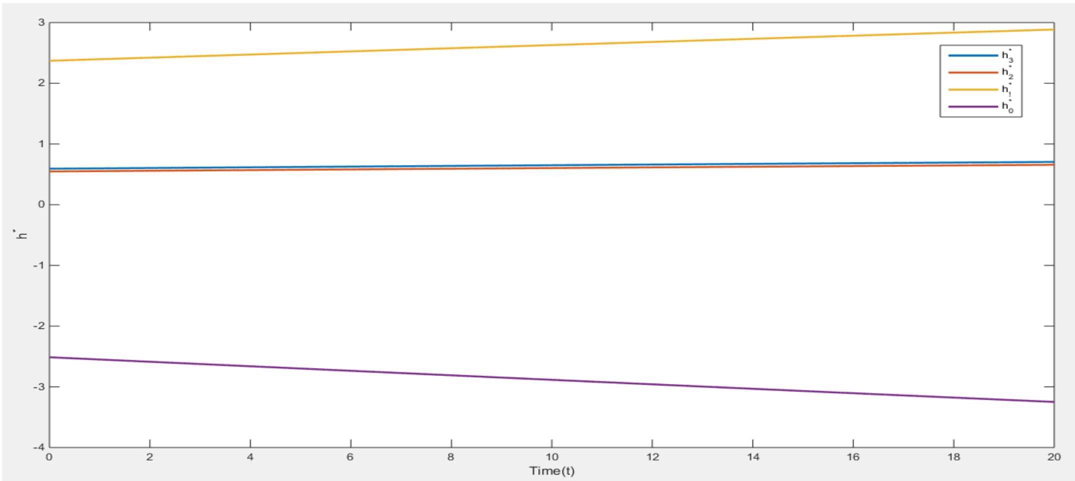

Fig. 3. Optimal investment plan with uneven distribution of remaining accumulated wealth when $q_t = q_0$

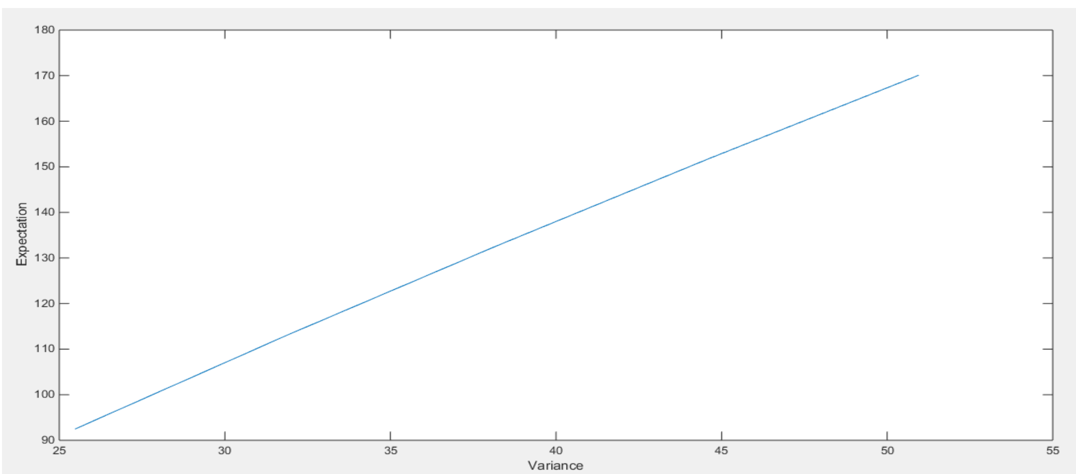

Fig. 4. Relationship between expectation and variance (Efficient Frontier)

5 Discussion

From the plots above, Fig. 1 shows the plot of optimal investment plan for the four assets with respect to time. We observed that the fund manager will invest more in the risky assets at the early stage of the accumulation period and subsequently reduce with time while investing a small fraction of its wealth in the risk free. Also as retirement age draws near, the fund manager reduces the fraction invested in risky assets while continuously increasing that of risk free asset. Also from Figs. 1 and 3, we observed that the proportion invested in two of the three risky asset were almost equal. Remark1 gives the optimal investment plan when the remaining accumulated wealth are equally distributed and this show that there is a disparity between the two plans for the two cases. Comparing Figs. 1 and 2, we observed a very different behaviour in investment policy of the fund managers when the remaining accumulations are equally distributed and they are not equally distributed.

In Fig. 3, we observed that investment in risk free asset decreases with time while that of the risky assets increases with time. This is because the fund manager started the investment with the initial wealth and not the optimal fund size as in the case of Figs. 1 and 2. Finally, Fig. 4, gives a relationship between the expectation and variance; which shows that members who are willing to take more risks stand a chance of having more at the end of the accumulation period and vice versa.

6 Conclusion

We investigated the optimal investment plan for a pension plan member with return of premium together with interest and uneven distribution of remaining accumulated wealth by surviving members. The portfolio was made up of four different assets which include a risk free asset and three risky assets whose prices were modelled by geometric Brownian motion. A stochastic optimal control problem was formulated and the resultant optimization problem was solved for the optimal investment plan and the efficient frontier, we present numerical simulations of the optimal investment plan with time using matlab software. We compare our result with a case where the remaining accumulations were equally distributed and conclude that the investment plans for the two cases are totally different.

Competing Interests

Authors have declared that no competing interests exist.

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