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Introducing the Clique-Safe Domination in Graphs

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Let $G = (V(G), E(G))$ be any finite, undirected, simple graph. A set $D \subseteq V(G)$ is introduced in this paper as a clique-safe dominating set of G if D is a dominating set of G and for every clique D'_m of size m in the subgraph induced by $V(G) \setminus D$, there exists a clique D_n of size n in the subgraph induced by D such that $n \geq m$. The clique-safe domination number of G, denoted by $\gamma_{cs}(G)$, is the smallest cardinality of a clique-safe dominating set of G. This study aims to generate a few elementary properties of the parameter and to characterize the minimum clique-safe dominating sets of paths and cycles. As a consequence, the clique-safe domination numbers of the aforesaid graphs are obtained.

Keywords: Clique-safe dominating set; clique-safe domination number; clique-safe; clique.

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1 Introduction

Let $G = (V(G), E(G))$ be a nontrivial graph which is finite, simple, and undirected. A nonempty subset D of $V(G)$ is a dominating set of G if for every vertex $y \in V(G) \setminus D$, there exists $x \in D$ such that $xy \in E(G)$. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set D of G with $|D| = \gamma(G)$ is called a minimum dominating set of G or a γ-set of G. Consider the following examples:

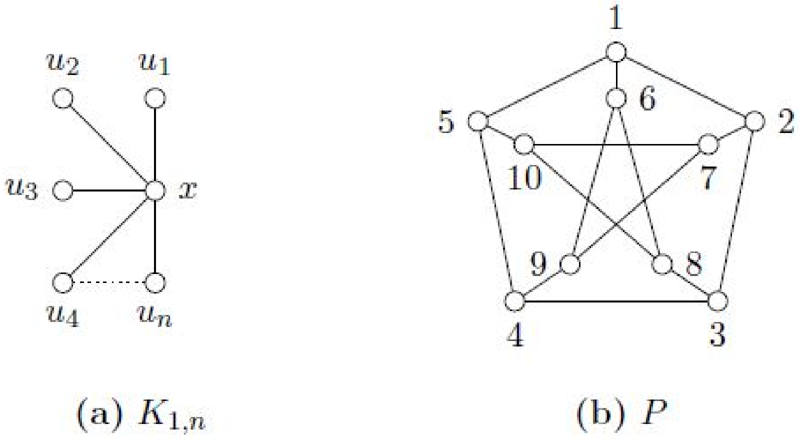


Fig. 1. The Star graph $K_{1,n}$ and the Petersen graph P

Example 1.1. *Notice that vertex x in Fig. 1a is adjacent to every other vertex in the star graph. Hence, $D = \{x\}$ is a γ-set of G and $\gamma(G) = 1$.*

Example 1.2. *Consider the Petersen graph P in Fig. 1b. Notice that $D_1 = \{1, 2, 3, 4, 5\}$ is a dominating set of G but not a γ-set of G since we can find a dominating set of G with fewer elements, say $D_2 = \{1, 8, 9\}$. The fact that D_2 in this case has three elements immediately means that $\gamma(G) \leq 3$. Since by simple inspection P does not have a dominating set of cardinality 1 or 2, it follows right away that $\gamma(P) \geq 3$. Combining the two inequalities produces $\gamma(P) = 3$, with the set $D_2 = \{1, 8, 9\}$ as a γ- set of P.*

Several types of dominating sets have been investigated, including independent dominating sets, total dominating sets, and connected dominating sets, each of which has a corresponding graphical parameter. In [1], Cockayne and Hedetnieme presented a quick review of results and applications concerning dominating sets in graphs. They have seen that the theory of domination resembles the well-known theory of graph colorings. Dominating sets have applications in a variety of fields, including communication theory and political science.

A dominating set D of G is defined in this paper as a clique-safe dominating set if the subgraph induced by D contains a clique D_n of greater cardinality than that of any clique D'_m in the subgraph induced by $V(G) \setminus D = D^c$. The minimum cardinality obtainable from among all clique-safe dominating sets of G is referred to as the clique-safe domination number of G, denoted by $\gamma_{cs}(G)$. Any clique-safe dominating set W of G such that $|W| = \gamma_{cs}(G)$ is called a minimum clique-safe dominating set of G or a γ_{cs}-set of G.

Example 1.3. *Consider the graph G in Fig. 2. Let $D = \{v_2, v_4\}$. Observe that D is a dominating set of G and $\langle D^c \rangle_G$ is a path of order 2. This means that the largest clique in $\langle D^c \rangle_G$ is of size 2 which is also the size of the largest clique in $\langle D \rangle_G$. Hence, D is a clique-safe dominating set of G. Since there is no singleton subset of $V(G)$ that will satisfy the definition of a clique-safe dominating set, it follows now that $\gamma_{cs}(G) = 2$.*

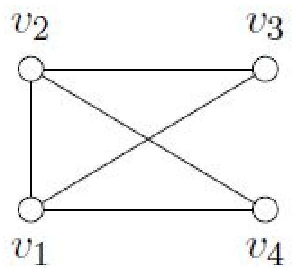


Fig. 2. The graph G

In this study, we shall generate some few elementary properties of the clique-safe dominating sets and characterize the minimum clique-safe dominating sets of the path P_n and cycle C_n. We shall also obtain the corresponding formulas for the clique-safe domination numbers of these parameterized families of graphs.

The concept of clique domination was first studied by Cozzens and Kelleher in [2]. A clique dominating set or a dominating clique is a dominating set that induces a complete subgraph. In their paper, they characterized the classes of graphs containing some dominating sets that induce complete subgraphs. In [3], Canoy and Daniel characterized the clique dominating sets in the join, corona, composition and cartesian product of graphs. Dominating sets that induce complete subgraphs have great diversity of applications. In addition, the properties of dominating sets are useful in identifying some structural properties of social networks. Domination and other variations of domination can be found in [4] and [5].

Throughout this paper, every graph is considered in the context of being simple, finite and undirected. Other standard terminologies not defined in this paper can be found in [6].

2 Main Results

Our first general observations are given below:

2.1 Some basic properties of the clique-safe domination number

Theorem 2.1. *The following properties hold for any graph G :*

a.) $1 \leq \gamma(G) \leq \gamma_{cs}(G)$;

b.) $\gamma_{cs}(G) = 1$ *if and only if G is the star $K_{1,n}$.*

Proof. Note that if ζ_1 is the collection of all dominating sets of G and ζ_2 is the collection of all clique-safe dominating sets of G, then ζ_1 is a superset of ζ_2. As an immediate consequence, we have $\gamma(G) \leq \gamma_{cs}(G)$, where the inequality $\gamma(G) \geq 1$ is also clear. Part (b) is straightforward. □

Remark 2.1. We conjecture at the moment that if G is any connected graph of order n, then $\gamma_{cs}(G) \leq \frac{n}{2}$.

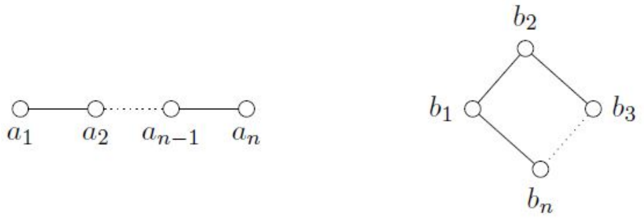
Theorem 2.2. *Let a and n be positive integers such that $1 \leq a \leq n$. Then there exists a graph G of order n such that $\gamma_{cs}(G) = a$.*

Proof. If $a = n$, then take G to be the null graph of order n and we are done. So suppose $a < n$. In this case, take G to be the disjoint union of $a - 1$ isolated vertices together with the star $K_{1,n-a}$. Clearly, G here is of order n such that $\gamma_{cs}(G) = a$. □

In what follows are the results characterizing the clique-safe dominating sets in paths and cycles.

2.2 Clique-Safe domination in paths and cycles

The path P_n of order n is the graph that may be drawn with distinct vertices $a_1, a_2, a_3, ..., a_n$ and edges $a_1a_2, a_2a_3, ...,$ and $a_{n-1}a_n$, while the cycle C_n of order $n \geq 3$ is the graph that may be drawn with distinct vertices $b_1, b_2, b_3, ..., b_n$ and edges $b_1b_2, b_2b_3, ..., b_{n-1}b_n$, and b_nb_1.



(a) Path P_n (b) Cycle C_n

Fig. 3. The path P_n and the cycle C_n

For the path P_n with $V(P_n) = \{a_1, a_2, a_3, ..., a_n\}$ and $E(P_n) = \{a_1a_2, a_2a_3, ..., a_{n-1}a_n\}$, where $n \geq 1$, suppose D is the minimum clique-safe dominating set of P_n. Then D takes the following forms for $n = 1, 2, 3, 4, 5, 6$:

- For $n = 1$: $D = \{a_1\}$, so that $\gamma_{cs}(P_1) = 1 = \gamma(P_1)$;
- For $n = 2$: $D = \{a_1\}$ or $D = \{a_2\}$, so that $\gamma_{cs}(P_2) = 1 = \gamma(P_2)$
- For $n = 3$: $D = \{a_2\}$, so that $\gamma_{cs}(P_3) = 1 = \gamma(P_3)$;
- For $n = 4$: $D = \{a_2, a_3\}$, or $D = \{a_1, a_3\}$ or $D = \{a_2, a_4\}$, so that $\gamma_{cs}(P_4) = 2 = \gamma(P_4)$;
- For $n = 5$: $D = \{a_2, a_4\}$, so that $\gamma_{cs}(P_5) = 2 = \gamma(P_5)$;
- For $n = 6$: The set $\{a_2, a_5\}$ is the unique minimum dominating set of P_6 , but not a clique-safe dominating set. This implies that $\gamma_{cs}(P_6) \geq 3$. Since we can choose a clique-safe dominating set $D = \{a_2, a_3, a_5\}$ or $D = \{a_2, a_4, a_5\}$ or $D = \{a_2, a_5, a_6\}$, etc., it follows that $\gamma_{cs}(P_6) \leq 3$. Combining the two inequalities, we obtain $\gamma_{cs}(P_6) = 3$.

Observe that for $n = 1, 2, 3, 4, 5$, $\gamma(P_n) = \gamma_{cs}(P_n)$, but for $n = 6$, $\gamma(P_6) \neq \gamma_{cs}(P_6)$. The following diagrams below provide more details.

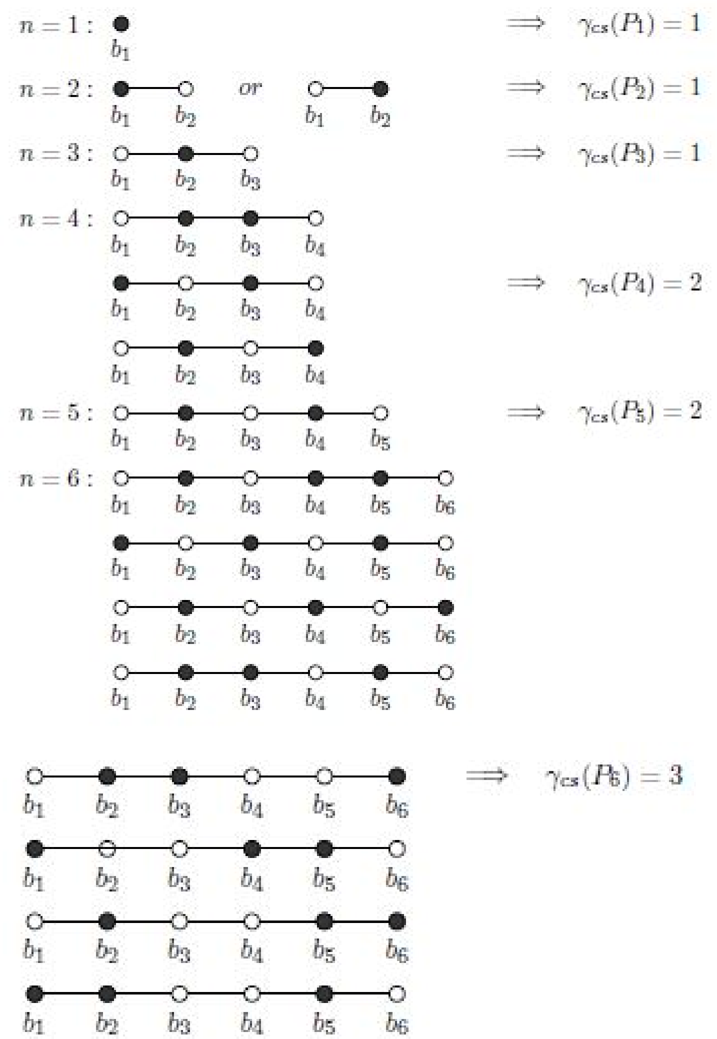
Our next results will make use of the concept of vertex covering of graphs. Recall that a vertex cover for a graph G is a set $S \subseteq V(G)$ such that for every edge $e \in E(G)$ there exists a vertex $x \in S$ such that x is an end-vertex of e. The vertex covering number of G denoted by $\alpha(G)$ is the minimum cardinality of such a vertex covering S of G.

Lemma 2.3. *[7] For the path P_n of order n, $\alpha(G) = \lfloor \frac{n}{2} \rfloor$.*

Theorem 2.4. *Let P_n be a path of order n, as described in Fig. 3a, where $n \geq 10$ and $n \in \{3k-1, 3k, 3k+1\}$ for some positive integer k. Let $\emptyset \subsetneq D \subseteq V(P_n)$. Then D is a minimum clique-safe dominating set of P_n if and only if D is a dominating set of P_n of minimum cardinality such that $|D| = k+1$ and the subgraph induced by D contains a K_2 subgraph.*

Proof. Consider 3 cases:

i. For $n = 3k$ for some $k \geq 4$:
Note first that the set $W = \{a_2, a_5, ..., a_{n-1}\}$ containing exactly k elements from $V(P_n) = \{a_1, a_2, a_3, ..., a_n\}$ is the unique γ-set of P_n, although not a clique-safe dominating set. As a consequence, $\gamma_{cs}(P_n) > k$. Now if D is a minimum clique-safe dominating set of P_n, then D is clearly dominating and $|D| > k$. Since



the set $W \cup \{a_n\}$ which contains $k+1$ elements is a clique-safe dominating set of P_n, it follows now that $|D| = k+1$. Since by Lemma 2.3 the minimum number of vertices needed to provide a covering of all the edges of P_n is equal to $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{3k}{2} \rfloor > k+1$ for $k \geq 4$, it follows that the subgraph induced by $V(P_n) \setminus D$ contains a K_2 subgraph. Thus, the subgraph induced by D also contains a K_2-subgraph.

ii. For $n = 3k+1$ for some $k \geq 3$:
Any subset $W \subseteq V(P_n)$ with at most k elements cannot dominate the path $P_n = [a_1, a_2, ..., a_n]$. This means that $\gamma(P_n) > k$ and, by Theorem 2.1(a), $\gamma_{cs}(P_n) > k$. So if D is a minimum clique-safe dominating set of P_n, then D is clearly a dominating set and $|D| > k$. Since the set $W = \{a_2, a_5, a_8, ..., a_{n-2}, a_{n-1}\}$ which contains $k+1$ elements is a clique-safe dominating set of P_n, it follows that $|D| = k+1$. Since by Lemma 2.3 the minimum number of vertices needed to provide a covering of all the edges of P_n is equal to $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{3k+1}{2} \rfloor > k+1$ for $k \geq 3$, it follows that the subgraph induced by $V(P_n) \setminus D$ contains a K_2 subgraph. Thus, the subgraph induced by D also contains a K_2-subgraph.

iii. For $n = 3k - 1$ for some $k \geq 4$:
Any subset $W \subseteq V(P_n)$ with at most k elements cannot clique-safe dominate the path P_n with $V(P_n) = \{a_1, a_2, ..., a_n\}$. So if D is a minimum clique-safe dominating set of P_n, then we have D dominating and $|D| > k$. Since the set $W = \{a_2, a_5, a_8, ..., a_{3k-4}, a_{3k-2}, a_{3k-1}\}$ which contains $k+1$ elements is a clique-safe dominating set of P_n, it follows that $|D| = k+1$. Since by Lemma 2.3 the minimum number of vertices needed to provide a covering of all the edges of P_n is equal to $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{3k-1}{2} \rfloor > k+1$ for $k \geq 4$, it follows that the subgraph induced by $V(P_n) \setminus D$ contains a K_2 subgraph. Thus, the subgraph induced by D also contains a K_2-subgraph.

The converse is straightforward. □

Corollary 2.5. *The clique-safe domination number of the path P_n is given by the following expression:*

$$\gamma_{cs}(P_n) = \begin{cases} 1 & \text{if } n = 1, 2, 3 \\ 2 & \text{if } n = 4, 5 \\ 3 & \text{if } n = 6, 7 \\ k+1 & \text{if } n \geq 8 \text{ and } n \in \{3k-1, 3k, 3k+1\} \text{ for some positive integer } k \end{cases} \tag{2.1}$$

Proof. The additional instances $\gamma_{cs}(P_7) = 3$, $\gamma_{cs}(P_8) = \gamma_{cs}(P_9) = 4$ can be verified analogously. For $n \geq 10$ with $n \in \{3k, 3k+1, 3k-1\}$ for some positive integer k, Theorem 2.4 asserts that $\gamma_{cs}(P_n) = k+1$. □

For the cycle C_n with $V(C_n) = \{b_1, b_2, b_3, ..., b_n\}$ and $E(C_n) = \{b_1b_2, b_2b_3, ..., b_nb_1\}$, where $n \geq 3$, suppose D is the minimum clique-safe dominating set of C_n. Then D takes the following forms for $n = 3, 4, 5$:

- For $n = 3$: $D = \{b_1, b_2\}$ or $D = \{b_1, b_3\}$ or $D = \{b_2, b_3\}$, so that $\gamma_{cs}(C_3) = 2$;
- For $n = 4$: $D = \{b_1, b_2\}$ or $D = \{b_1, b_3\}$ or $D = \{b_1, b_4\}$ or $D = \{b_2, b_3\}$ or $D = \{b_2, b_4\}$ or $D = \{b_3, b_4\}$, so that $\gamma_{cs}(C_4) = 2$
- For $n = 5$: $D = \{b_1, b_2, b_3\}$ or $D = \{b_1, b_2, b_4\}$ or $D = \{b_1, b_2, b_5\}$ or $D = \{b_1, b_3, b_4\}$ or $D = \{b_1, b_3, b_5\}$ or $D = \{b_1, b_4, b_5\}$ or $D = \{b_2, b_3, b_4\}$ or $D = \{b_2, b_3, b_5\}$ or $D = \{b_2, b_4, b_5\}$ or $D = \{b_3, b_4, b_5\}$, so that $\gamma_{cs}(C_5) = 3$;

For $n = 6, 7, 8$, the clique safe domination number can be obtained analogously, where $\gamma_{cs}(P_6) = \gamma_{cs}(P_7) = 3$ and $\gamma_{cs}(P_8) = 4$.

Theorem 2.6. *Let C_n be a cycle of order n as described in Fig. 3b, where $n \geq 9$ and $n \in \{3k-1, 3k, 3k+1\}$ for some positive integer k. Let $D \subseteq V(C_n)$ be a nonempty set. Then D is a minimum clique-safe dominating set of C_n if and only if D is a dominating set of C_n of minimum cardinality such that $|D| = k+1$ and the subgraph induced by D contains a K_2 subgraph.*

Proof. Consider 3 cases:

i. For $n = 3k$ for some $k \geq 3$:
Note first that the sets $W_1 = \{b_1, b_4, ..., b_{n-2}\}$, $W_2 = \{b_2, b_5, ..., b_{n-1}\}$, and $W_3 = \{b_3, b_6, ..., b_n\}$ containing exactly k elements each from $V(C_n) = \{b_1, b_2, b_3, ..., b_n\}$ are the γ-sets of C_n, but none of which is a clique-safe dominating set. As a consequence, $\gamma_{cs}(C_n) > k$. Now if D is a minimum clique-safe dominating set of C_n, then D is clearly dominating and $|D| > k$. Note that adding another vertex to the set W_i for $i = 1, 2, 3$ to form W_i^* would make W_i^* a clique-safe dominating set of C_n, where $|W_i^*| = k+1$. Since the minimum number of vertices needed to provide a covering of all the edges of C_n is equal to $\lceil \frac{n}{2} \rceil = \lceil \frac{3k}{2} \rceil > k+1$ for $k \geq 3$, it follows that the subgraph induced by $V(C_n) \setminus D$ contains a K_2 subgraph. Thus, the subgraph induced by D also contains a K_2-subgraph.

ii. For $n = 3k + 1$ for some $k \geq 3$:
Any subset $W \subseteq V(C_n)$ with at most k elements cannot dominate the cycle C_n with $V(C_n) = \{b_1, b_2, ..., b_n\}$. This means that $\gamma(C_n) > k$ and, by Theorem 2.1(a), $\gamma_{cs}(C_n) > k$. So if D is a minimum clique-safe dominating set of C_n, then D is clearly a dominating set and $|D| > k$. Since the set $W^* = \{b_1, b_4, ..., b_{n-3}, b_{n-2}\}$ containing exactly $k + 1$ elements is a clique-safe dominating sets of C_n, it follows that $|D| = k + 1$. Since the minimum number of vertices needed to provide a covering of all the edges of C_n is equal to $\lceil \frac{n}{2} \rceil = \lceil \frac{3k+1}{2} \rceil > k + 1$ for $k \geq 3$, it follows that the subgraph induced by $V(C_n) \setminus D$ contains a K_2 subgraph. Thus, the subgraph induced by D also contains a K_2-subgraph.

iii. For $n = 3k - 1$ for some $k \geq 4$:
Any subset $W \subseteq V(C_n)$ with at most k elements cannot clique-safe dominate the cycle C_n with $V(C_n) = \{b_1, b_2, b_3, ..., b_n\}$. So if D is a minimum clique-safe dominating set of C_n, then we have D dominating and $|D| > k$. Since the set $W^* = \{b_2, b_5, b_8, ..., b_{n-1}, b_n\}$ which contains $k + 1$ elements is a clique-safe dominating set of C_n, it follows that $|D| = k + 1$. Since the minimum number of vertices needed to provide a covering of all the edges of C_n is equal to $\lceil \frac{n}{2} \rceil = \lceil \frac{3k-1}{2} \rceil > k + 1$ for $k \geq 4$, it follows that the subgraph induced by $V(C_n) \setminus D$ contains a K_2 subgraph. Thus, the subgraph induced by D also contains a K_2-subgraph.

The converse is straightforward. □

Corollary 2.7. *The clique-safe domination number of cycle C_n where $n \geq 9$ and $n \in \{3k - 1, 3k, 3k = 1\}$ for some positive integer k, is given by $\gamma_{cs}(C_n) = k + 1$.*

Proof. This is a direct consequence of Theorem 2.6. □

3 Conclusion

In this article the concept of clique-safe domination is introduced and its corresponding parameter clique-safe domination number investigated. Furthermore, the corresponding expressions for the clique-safe domination number of the paths and cycles are determined for some specific orders. Finally, the parameter introduced in this paper may be explored further to address some relevant problems as done in [8], [9], [10], [11], [12], , and [13].

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Competing Interests

Authors have declared that no competing interests exist.

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