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Fixed Point Theorems Using α −Admissible Mappings in Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we shall prove the fixed point theorems in metric space using α −admissible mapping. Some existing results of literature shall be deduced from the main results. In the end, we shall provide an example to support our result.

Keywords: α −admissible mappings; complete metric space; fixed point.

2010 MSC: 47H10, 54H25.

1 Introduction

In 1922, Banach gave a principle to obtain the fixed point in the complete metric space. Since then, many researchers have worked on the Banach fixed point theorem (see, for example, [1-38]) and tried to generalize this principle. In 2012, Samet et al. [25] introduced the new concepts of mappings called α −admissible mappings in metric space. Recently, in 2013 Farhan et al. [1] gave new contractions using α −admissible mapping in metric spaces.

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In this paper, we shall generalize Farhan's et al. [1] contractions and give fixed point theorems for such contractions.

2 Preliminaries

To prove our main results we need some basic definitions from literature as follows:

Definition([39]): "Let X be a set. A metric space is an ordered pair (X,d) where d is a function $d: X \times X \rightarrow \mathbb{R}$ such that

(1) $d(x,y) \geq 0.$ (non-negativity)
(2) $d(x,y) = 0$ iff $x = y.$ (non-degeneracy)
(3) $d(x,y) = d(y,x)$ (symmetry)
(4) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)"

Definition([40]) : "A sequence $\{x_n\}$ in a metric space (X,d) is said to converge if there is a point $x \in X$ and for every $\in > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \in$ for every $n > N$".

Definition: "A sequence $\{x_n\}$ in a metric space (X,d) is Cauchy if for every $\in > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \in$ for every $n, m > N$".

Definition([40]) : "A metric space $(X,d)$is said to be complete if every Cauchy sequence is convergent".

Definition([25]) : "Let $f: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0,\infty)$. We say that f is an $\alpha -$admissible mapping if $\alpha(x,y) \geq 1$ implies $\alpha(fx,fy) \geq 1$, for all $x,y \in X$".

3 Main Results

In this section, we shall prove fixed point theorems.

Theorem 3.1: Let (X,d) be a complete metric space and $T: X \rightarrow X$ be an $\alpha -$ admissible mapping. Assume that there exists a function $\beta: [0,\infty) \rightarrow [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(\mathrm{d}(\mathrm{Tx},\mathrm{Ty}) + \mathrm{l})^{\alpha(\mathrm{x},\mathrm{Tx})\alpha(\mathrm{y},\mathrm{Ty})} \leq \beta\big(\mathrm{M}(\mathrm{x},\mathrm{y})\big)\mathrm{M}(\mathrm{x},\mathrm{y}) + \mathrm{l}, \text{ for all } \mathrm{x},\mathrm{y} \in \mathrm{X} \text{ and } \mathrm{l} \geq 1. \quad (1)$$

Where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$

Suppose that either

(3.1) T is continuous, or

(3.2) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$, for all n, then $(\alpha, Tx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.

If $x_{n+1} = x_n$, for some $n \in N$, then $Tx_n = x_n$ and we are done.

So, we suppose that $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$.

Since T is $\alpha -$admissible, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ which implies $\alpha(x_0, x_1) \geq 1$.

Similarly, we can say that $\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$.

By continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \quad (2)$$

By using equation (2), we have

$$\mathrm{d}(\mathrm{x_n}, \mathrm{x_{n+1}}) + \mathrm{l} = \mathrm{d}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}) + \mathrm{l} \leq (\mathrm{d}(\mathrm{Tx_{n-1}}, \mathrm{Tx_n}) + \mathrm{l})^{\alpha(\mathrm{x_{n-1}}, \mathrm{Tx_{n-1}})\alpha(\mathrm{x_n}, \mathrm{Tx_n})}.$$

Now using equation (1), we get

$$d(x_n, x_{n+1}) + l \leq \beta\big(M(x_{n-1}, x_n)\big)M(x_{n-1}, x_n) + l, \quad (3)$$

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\},$$
$$= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

Assume that if possible $d(x_n, x_{n+1)} > d(x_{n-1}, x_n)$.

Then, $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$.

Using this in equation (3), we get

$$d(x_n, x_{n+1}) < \beta\big(d(x_n, x_{n+1})\big)d(x_n, x_{n+1}) \quad (4)$$

$\Rightarrow d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

So $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, for all n.

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive real numbers. So, it is convergent and suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = d$. Clearly, $d \geq 0$.

Claim: $d = 0$.

Equation (4) implies that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n) \leq 1,$$

Which implies that $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n) = 1$.

Using the property of the function β, we conclude that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. (5)

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\in > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k, we have

$$n(k) > m(k) > k, d\big(x_{n(k)}, x_{m(k)}\big) \geq \in \text{ and } d\big(x_{n(k)}, x_{m(k)-1}\big) < \in.$$

By the triangle inequality, we have

$$\in \leq d\big(x_{n(k)}, x_{m(k)}\big) \leq d\big(x_{n(k)}, x_{m(k)-1}\big) + d(x_{m(k)-1}, x_{m(k)})$$
$$< \in + d\big(x_{m(k)-1}, x_{m(k)}\big), \text{ for all } k \in \mathbb{N}.$$

Taking the limit as $k \to +\infty$ in the above inequality and using equation (5), we get

$$\lim_{k\to+\infty} d\big(x_{n(k)}, x_{m(k)}\big) = \in. \quad (6)$$

Again by triangle inequality, we have

$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)})$.

Taking the limit as $k \to +\infty$, together with (5) and (6), we deduce that

$$\lim_{k \to +\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \in. \quad (7)$$

From equations (1),(2),(6) and (7), we get

$$\begin{aligned} d(x_{n(k)+1}, x_{m(k)+1}) + l &\leq (d(x_{n(k)+1}, x_{m(k)+1}) + l)^{\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})}, \\ &= (d(Tx_{n(k)}, Tx_{m(k)}) + l)^{\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})} \\ &\leq \beta(M(x_{n(k)}, x_{m(k)})M(x_{n(k)}, x_{m(k)}) + l \end{aligned} \quad (8)$$

$$M(x_{n(k)}, x_{m(k)}) = \max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1})\}$$

So, equation (8) implies that

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq \beta(M(x_{n(k)}, x_{m(k)})M(x_{n(k)}, x_{m(k)}) \leq 1,$$

Letting $k \to \infty$, we get

$$\lim_{k \to \infty} \beta(d(x_{n(k)}, x_{m(k)}) = 1.$$

By using definition of β function, we get

$\Rightarrow \lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = 0 < \in$, which is a contradiction.

Hence, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete space, so $\{x_n\}$ is convergent and assume that $x_n \to x$ as $n \to \infty$.

Since T is continuous, then we have

$$Tx = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x.$$

So, x is a fixed point of T.

Now, suppose that (3.2) holds, then $\alpha(x, Tx) \geq 1$ and by using equations (1) and (2) we get

$$d(Tx, x_{n+1}) + l \leq (d(Tx, Tx_n) + l)^{\alpha(x,Tx)\alpha(x_n,Tx_n)}$$

$$\leq \beta(M(x, x_n))M(x, x_n) + l. \quad (9)$$

Where $M(x, x_n) = \max\{d(x, Tx), d(x, x_n), d(x, Tx_n)\}$.

Clearly, from equation (9) and using triangle inequality, we get

$$\begin{aligned} d(Tx, x) &\leq d(Tx, x_{n+1}) + d(x_{n+1}, x) \\ &\leq \beta(M(x, x_n))M(x, x_n) + d(x_{n+1}, x) \end{aligned}$$

Letting $n \to \infty$, we get

$d(Tx, x) = 0$ which implies $Tx = x$.

Theorem 3.2: Let (X, d) be a complete metric space and $T: X \to X$ be an α −admissible mapping. Assume that there exists a function $\beta: [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ which implies that $t_n \to 0$ and

$$(\alpha(x, Tx).\alpha(y, Ty) + 1)^{d(Tx,Ty)} \leq 2^{\beta(M(x,y))M(x,y)}, \text{ for all } x, y \in X. \tag{10}$$

Suppose that either

(3.3) T is continuous or

(3.4) If $\{x_n\}$ is a sequence in X such that $x_n \to x, \alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Define a sequence $\{x_n\}$ in X as $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then $Tx_n = x_n$ and we are done. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. As in Theorem (3.1), we conclude that $\alpha(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N}$.

From equation (10), we get

$$2^{d(Tx_{n-1},Tx_n)} \leq (\alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Tx_n) + 1)^{d(Tx_{n-1},Tx_n)}$$
$$\leq 2^{\beta(M(x_{n-1},x_n))M(x_{n-1},x_n)}$$

Which yields that

$$d(x_n, x_{n+1}) \leq \beta\big(M(x_{n-1}, x_n)\big)M(x_{n-1}, x_n). \tag{11}$$

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\},$$
$$= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

If possible suppose that

$$d(x_n, x_{n+1}) > d(x_{n-1}, x_n).$$

Then $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$

Using this, equation (11) implies that $d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

So, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.

So, $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive reals. So, there exists $d \in R^+ \cup \{0\}$ such that

$$d(x_n, x_{n+1}) \to d \text{ as } n \to \infty$$

Claim: $d = 0$.

Equation (11) implies that

$\frac{d(x_n,x_{n+1})}{M(x_{n-1},x_n)} \leq M\big(d(x_{n-1}, x_n)\big) \leq 1$, which implies

$\frac{d(x_n,x_{n+1})}{d(x_{n-1},x_n)} \leq \beta\big(d(x_{n-1}, x_n)\big) \leq 1$.

Taking $n \to \infty$, we get $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n) = 1$. Using definition of β function, we get

$$\Rightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{12}$$

We prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\in > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k,

$$n(k) > m(k) > k, d\big(x_{n(k)}, x_{m(k)}\big) \geq \in \text{ and } d\big(x_{n(k)}, x_{m(k)-1}\big) < \in.$$

Following the related lines in the proof of Theorem (3.1), we get

$$\lim_{k\to+\infty} d\big(x_{n(k)}, x_{m(k)}\big) = \in \tag{13}$$

$$\text{and } \lim_{k\to+\infty} d\big(x_{n(k)+1}, x_{m(k)+1}\big) = \in. \tag{14}$$

Using equations (10), (13) and (14), we get

$$\begin{aligned} 2^{d(x_{n(k)+1}, x_{m(k)+1})} &\leq \big(\alpha\big(x_{n(k)}, Tx_{n(k)}\big)\alpha\big(x_{m(k)}, Tx_{m(k)}\big) + 1\big)^{d(x_{n(k)+1}, x_{m(k)+1})} \\ &= \big(\alpha\big(x_{n(k)}, Tx_{n(k)}\big)\alpha\big(x_{m(k)}, Tx_{m(k)}\big) + 1\big)^{d(Tx_{n(k)}, Tx_{m(k)})} \\ &\leq 2^{\beta(M(x_{n(k)}, x_{m(k)}))M(x_{n(k)}, x_{m(k)})} \end{aligned}$$

Where $M\big(x_{n(k)}, x_{m(k)}\big) = \max\big\{d\big(x_{n(k)}, x_{m(k)}\big), d\big(x_{n(k)}, x_{n(k)+1}\big), d\big(x_{m(k)}, x_{m(k)+1}\big)\big\}$.

Hence, $\frac{d(x_{n(k)+1}, x_{m(k)+1})}{M(x_{n(k)}, x_{m(k)})} \leq \beta\big(M\big(x_{n(k)}, x_{m(k)}\big) \leq 1$.

By taking $k \to \infty$, we get

$$\lim_{k\to\infty} \beta\big(d\big(x_{n(k)}, x_{m(k)}\big) = 1.$$

$\Longrightarrow \lim_{k\to\infty} d\big(x_{n(k)}, x_{m(k)}\big) = 0 < \in$, which is a contradiction.

So, $\{x_n\}$ is a Cauchy sequence and as X is complete, so $\{x_n\} \to x$.

Now suppose the T is continuous.

$$\mathrm{Tx} = \lim_{n\to\infty} \mathrm{Tx_n} = \lim_{n\to\infty} \mathrm{x_{n+1}} = \mathrm{x}.$$

$\Rightarrow Tx = x$.
$\Rightarrow x$ is fixed point of T.

Next, we suppose that the condition (3.4) holds, then $\alpha(x, Tx) \geq 1$.

Now by equation (10), we get

$$\begin{aligned} 2^{d(Tx, x_{n+1})} &\leq (\alpha(x, Tx)\alpha(x_n, Tx_n) + 1)^{d(Tx, Tx_n)}, \\ &\leq 2^{\beta(M(x, x_n))M(x, x_n)}, \end{aligned}$$

Where

$$\mathrm{M(x, x_n) = \max\{d(x, x_n), d(x, Tx), d(x_n, x_{n+1})\}}$$

$$\Rightarrow d(Tx, x_{n+1}) \leq \beta\big(M(x, x_n)\big)M(x, x_n) \tag{15}$$

Using triangle inequality,

$$d(Tx, x) \leq d(Tx, x_{n+1}) + d(x_{n+1}, x).$$

Letting $n \to \infty$ and using (15), we get

$$d(Tx, x) = 0.$$
$$\Rightarrow Tx = x.$$

Theorem 3.3: Let (X, d) be a complete metric space and $T: X \to X$ be an α −admissible mapping. Assume that there exists a function $\beta: [0, \infty) \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and

$$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq \beta(M(x, y))M(x, y), \forall x, y \in X. \quad (16)$$

Suppose that either

(3.5) T is continuous or

(3.6) If $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, Tx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ $\forall n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $Tx_n = x_n$ and we are done.

So, assume $x_n \neq x_{n+1}$ for all $n \geq 1$.

As in Theorem (3.1), we conclude that $\alpha(x_n, x_{n+1}) \geq 1$ for all n. (17)

Now by equation (16), we get

$$d(x_n, x_{n+1}) \leq \alpha(x_n, Tx_{n-1})\alpha(x_n, Tx_n)d(Tx_{n-1}, Tx_n) \leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n). \quad (18)$$

Where $M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)\}$.
$= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$

If possible assume that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$.

Using this from equation (16), we get

$$d(x_n, x_{n+1}) \leq \beta(d(x_{n-1}, x_n)d(x_{n-1}, x_n) \quad (19)$$

$d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

So, $d(x_n, x_{n+1}) \leq d(x_n, x_{n-1})$for all $n \in \mathbb{N}$.

It follows that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive reals. So, there exists $d \geq 0$ such that $d(x_n, x_{n+1}) \to d$ as $n \to \infty$.

Therefore, (19) implies that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) \leq 1.$$

Thus we find that $\lim_{n \to \infty} \beta(d(x_n, x_{n+1})) = 1$.

$$\Rightarrow \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \quad (20)$$

Next, we will prove that the sequence $\{x_n\}$ is Cauchy. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\in > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k,

$$n(k) > m(k) > k, d(x_{n(k)}, x_{m(k)}) \geq \in \text{ and } d(x_{n(k)}, x_{m(k)-1}) < \in.$$

Again, by following the lines of the proof of Theorem (3.1), we derive that

$$\lim_{k \to +\infty} d(x_{n(k)}, x_{m(k)}) = \in. \quad (21)$$

$$\text{and } \lim_{k \to +\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \in. \quad (22)$$

now, combining (16), (21) and (22), we get

$$\begin{aligned} d(x_{n(k)+1}, x_{m(k)+1}) &\leq \alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})d(x_{n(k)+1}, x_{m(k)+1}) \\ &= \alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \beta(M(x_{n(k)}, x_{m(k)})M(x_{n(k)}, x_{m(k)}). \end{aligned} \quad (23)$$

Where $M(x_{n(k)}, x_{m(k)}) = \max\{d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{m(k)})\}$

Now, equation (23) implies

$$\frac{d(x_{n(k)+1}, x_{m(k)+1})}{M(x_{n(k)}, x_{m(k)})} \leq \beta\left(M(x_{n(k)}, x_{m(k)})\right) \leq 1.$$

Taking $\to \infty$, we get

$$\lim_{k \to \infty} \beta(d(x_{n(k)}, x_{m(k)}) = 1.$$

$\Rightarrow \lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) = 0 < \in$, which is a contradiction.

So, $\{x_n\}$ is a Cauchy sequence. Since X is complete, so $\{x_n\} \to x$.

First suppose that T is continuous. So,

$$Tx = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = x.$$

$\Rightarrow x$ is fixed point.

Now suppose that (3.6) holds, then $\alpha(x, Tx) \geq 1$ and using (16), we have

$$\begin{aligned} d(Tx, x_{n+1}) &\leq d(Tx, Tx_n)\alpha(x, Tx)\alpha(x_n, Tx_n) \\ &\leq \beta(M(x, x_n))M(x, x_n) \end{aligned} \quad (24)$$

Where $M(x, x_n) = \max\{d(x, x_n), d(x, Tx), d(x_n, Tx_n)\}$

and $d(Tx, x) \leq d(Tx, x_{n+1}) + d(x_{n+1}, x)$

Using (24), we get

$$d(Tx, x) \leq \beta(M(x, x_n))M(x, x_n) + d(x_{n+1}, x)$$

Taking as $n \to \infty$, we get

$$d(Tx, x) = 0 \Rightarrow Tx = x.$$

Theorem 3.4: Assume that all the hypothesis of theorem (3.1), (3.2) and (3.3) hold. Adding the following condition:

(3.7) If $x = Tx$, then $\alpha(x, Tx) \geq 1$.

We obtain the uniqueness of fixed point.

Proof: Let z and z^* be two distinct fixed point of T in the setting of Theorem (3.1) and condition (3.7) holds, then

$\alpha(z, Tz) \geq 1$ and $\alpha(z^*, Tz^*) \geq 1$.

$$\text{So, } d(Tz, Tz^*) + l \leq (d(Tz, Tz^*) + l)^{\alpha(z,Tz)\alpha(z^*,Tz^*)}$$

$$\leq \beta(M(z, z^*))M(z, z^*) + l. \quad (25)$$

Where $M(z, z^*) = \max\{d(z, z^*), d(Tz, z), d(Tz^*, z)\}$
$= d(z, z^*)$.

So, equation (25) implies

$$d(z, z^*) = d(Tz, Tz^*) \leq \beta(d(z, z^*))d(z, z^*)$$
$$\Rightarrow \beta(d(z, z^*)) = 1$$
$$\Rightarrow d(z, z^*) = 0 \Rightarrow z = z^*.$$

Similarly, one can prove for theorem (3.2) and (3.3).

Example 3.5: Let $X = \{0, 1, 2\}$ and $d(x, y) = |x - y|$. Clearly, (X, d) is a complete metric space.

Define $T(0) = 0, T(1) = 0$ and $T(2) = 2$.

All possible pairs of (x, y) are as follows:

(x, y)	$d(Tx, Ty)$	$d(x, Tx)$	$d(y, Ty)$	$d(x, y)$	$M(x, y)$
(0, 0)	0	0	0	0	0
(0, 1)	0	0	1	1	1
(0, 2)	2	0	0	2	2
(1, 1)	0	1	1	0	1
(1, 2)	2	1	0	1	1
(2, 2)	0	0	0	0	0

Let $\alpha = 1$ and $\beta = \frac{1}{2}$.

Putting these values in equation (1), we get

When $(x, y) = (0, 0)$, $(0 + l) \leq \frac{1}{2}(0) + l \Rightarrow l \leq l$.

When $(x, y) = (0, 1)$, $(0 + l) \leq \frac{1}{2}(1) + l \Rightarrow l \leq \frac{1}{2} + l$.

When $(x, y) = (0, 2)$, $(2 + l) \leq \frac{1}{2}(2) + l \Rightarrow 2 + l \leq 1 + l$.

When $(x, y) = (1, 1)$, $(0 + l) \leq \frac{1}{2}(1) + l \Rightarrow l \leq \frac{1}{2} + l$.

When $(x, y) = (1, 2), (2 + l) \leq \frac{1}{2}(1) + l \Rightarrow 2 + l \leq \frac{1}{2} + l.$

When $(x, y) = (2, 2), (0 + l) \leq \frac{1}{2}(0) + l \Rightarrow l \leq l.$

Hence, theorem (3.1) is verified.

4 Consequences

Some existing results of literature can be deduced from our main results as follows:

Corollary 4.1.(Farhan et al. [1]) Let (X, d) be a complete metric space and $f: X \to X$ be an α −admissible mapping. Assume that there exists a function $\beta: [0, \infty) \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and

$$(d(fx, fy) + l)^{\alpha(x,fx)\alpha(y,fy)} \leq \beta(d(x, y))d(x, y) + l$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either

(a) f is continuous, or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to x, \alpha(x_n, x_{n+1}) \geq 1$ for all n, then $(\alpha, fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 1, one can get the proof.

Corollary 4.2. (Farhan et al. [1]) Let (X, d) be a complete metric space and $f: X \to X$ be an α −admissible mapping. Assume that there exists a function $\beta: [0, \infty) \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and

$$(\alpha(x, fx)\alpha(y, fy) + 1)^{d(fx,fy)} \leq 2^{\beta(d(x,y))d(x,y)}$$

for all $x, y \in X$. Suppose that either

(a) f is continuous, or
(b) If $\{x_n\}$ is a sequence in X such that $x_n \to x, \alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 2.

Corollary 4.3. (Farhan et al. [1]) Let (X, d) be a complete metric space and $f: X \to X$ be an α −admissible mapping. Assume that there exists a function $\beta: [0, \infty) \to [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and

$$\alpha(x, fx)\alpha(y, fy)d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$. Suppose that either

(a) f is continuous, or
(b) If $\{x_n\}$ is a sequence in X such that $x_n \to x, \alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 3, one can get the proof easily.

Corollary 4.4. (Farhan et al. [1]) Assume that all the hypotheses of theorem (1), (2) and (3) hold. Adding the following condition:

(c) If $x = fx$, then $\alpha(x, fx) \geq 1$,

we obtain the uniqueness of the fixed point of f.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 4.

Competing Interests

Authors have declared that no competing interests exist.

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