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On Synchronization of Pinning-Controlled Networks with Reducible and Asymmetric Coupling Matrix

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Abstract

This paper investigates the synchronization of directed networks whose coupling matrices are reducible and asymmetrical by pinning-controlled schemes. A strong sufficient condition is obtained to guarantee that the synchronization of the kind of networks can be achieved. For the weakly connected network, a method is presented in detail to solve two challenging fundamental problems arising in pinning control of complex networks: 1) How many nodes should be pinned? 2) How large should the coupling strength be used in a fixed complex network to realize synchronization? Then, we show the answer to the question that why all the diagonal block matrices of Perron-Frobenius normal matrices should be pinned? Besides, we find out the relation between the Perron-Frobenius normal form of coupling matrix and the differences of two synchronization conditions for strongly connected networks and weakly connected ones with linear coupling configuration. Moreover, we propose adaptive feedback algorithms to make the coupling strength as small as possible. Finally, numerical simulations are given to verify our theoretical analysis.

Keywords: Globally Synchronization, Complex Networks, Pinning Control, Reducible and Asymmetric Matrix

1. Introduction

Complex dynamical networks are found to be common systems in our real world [1-6], such as genetic regulatory networks, biological neural networks, telephone graphs, etc. Therefore, lots of researchers from many fields of science and engineering are attracted to focus on analyzing their complex behaviors. Particularly, the synchronization of a complex network has received great interest and attention. In mathematical view, a concrete system can be modeled simply by a graph, where the nodes represent individuals of the networks and the edges stand for interactions between them. Therefore, Lyapunov stability theories and algebraic graph theories, as two essential tools, are used to study dynamic behaviors of the complex networks.

So far, as a basic and very useful kind of synchronization phenomenon, complete synchronization has been investigated widely [7-12]. For example, the authors studied the global synchronization for a collection of nonlinearly coupled chaotic systems with an asymmetrical coupling matrix in [7]. In [8], two profound problems were solved: one is how to choose suitable pinning schemes for a given complex network, the other is how large the coupling strength should be used in a complex network to achieve synchronization. Besides, [9] showed the number of node which should be pinned in a complex network in order to reach synchronization. It can be seen that in [13-17] that pinning adaptive method is very effective for solving synchronization of complex networks. In [16], the authors investigated the synchronization of nonlinearly coupled networks through an innovative local adaptive approach. By using the technique of pinning only a limited subset of the whole network, [17] showed that the complex network with coupled identical oscillators could be driven onto some desired common reference trajectory. Nowadays, more and more attentions are being paid to the optimization problems of pinning schemes [18-21]. For instance, the crucial problem that how to select an optimal combination between the number of pinned nodes and the feedback control gain is studied in [19].

However, the configuration coupling matrix is assumed to be symmetric and irreducible in most of previous literatures, which implies that the topology of the corresponding complex network is undirected and strongly connected. It is not consistent with the realistic world. For a strongly connected network, a single controller is enough to pin the network to the desired state. Whereas, in a weakly connected directed network, more than one nodes should be pinned to guarantee the network synchronization could be reached. Furthermore, the existent synchronization conditions usually require that the coupling strength must be large enough, which is not practical indeed. Thus, the interesting problem that how large coupling strength should be used for synchronization is raised. Until now, the fruitful results are rather few and sometimes it is hard to meet the requirement of practical operation.

In this paper, the constraints that the configuration coupling matrices are symmetric and irreducible are removed in order to overcome the aforementioned shortcomings arising from the constraints on the configuration coupling matrix. Instead, we assume the coupling matrix is asymmetric, weighted and reducible, which is more consistent with the realistic world, such as broadcasting. Moreover, the minimum number of pinned nodes and the quantity of coupling strength are obtained by an implicit inequality. And, an adaptive technique is adopted to make the coupling strength as small as possible.

The rest of this paper is organized as follows: In Section 2, some definitions, lemmas, hypotheses and basic models are proposed. In Section 3, based on Lyapunov stability theories, some criteria for the global and exponential synchronization in two cases are derived. The numerical simulations are given in Section 4, while some concluding remarks are displayed in Section 5.

We denote an n-order identity matrix by I_n throughout this paper. The symmetric part of $\boldsymbol{A}=\left(a_{ij}\right)\in R^{m\times n}$ is shown as $\boldsymbol{A}^s=\left(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}\right)/2$. Moreover, $\|\boldsymbol{A}\|_1=\max\limits_{1\le i\le m}\sum\limits_{j=1}^{n}\left|a_{ij}\right|$ and $\|\boldsymbol{A}\|_\infty=\max\limits_{1\le j\le n}\sum\limits_{i=1}^{m}\left|a_{ij}\right|$.

An *n*-order square matrix is positive definite denoted by $B>0$. If all eigenvalues of B are real, then we denote its k th largest eigenvalue as $\lambda_k(B)$. At last, the notation $\otimes$ is Kronecker product.

2. Preliminaries and Model Description

Consider a complex dynamical network consisting of N identical nodes with linearly diffusive coupling, which can be described by

$$\dot{x}_i(t)=f\left(x_i(t),t\right)+c\sum_{j=1,\,j\ne i}^{N}a_{ij}\Gamma\left(x_j(t)-x_i(t)\right),\ i=1,\cdots,N,$$

where $x_i(t)=\left[x_i^1(t),\cdots,x_i^n(t)\right]^T\in R^n$ is the state variable of the ith node, $t\in\left[0,+\infty\right)$ is the continuous time, $f:R^n\times\left[0,+\infty\right)\to R^n$ is a continuous map, c is the coupling strength, and $\Gamma\in R^{n\times n}$ is the inner coupling matrix. $\boldsymbol{A}=\left(a_{ij}\right)_{N\times N}$ is the coupling configuration matrix representing the topological structure of the network, where a_{ij} is defined as follows: if there exists a connection between node i and node j, then $a_{ij}>0$, otherwise, $a_{ij}=0$ $(j\ne i)$. Then the diagonal elements of matrix $\boldsymbol{A}$ are given by $a_{ii}=-\sum\limits_{j=1,\,j\ne i}^{N}a_{ij}$, which ensures the diffusion that $\sum\limits_{j=1}^{N}a_{ij}=0$. Equivalently, network (1) can be rewritten in a simpler form as follows:

$$\dot{x}_i(t)=f\left(x_i(t),t\right)+c\sum_{j=1}^{N}a_{ij}\Gamma x_j(t),\ i=1,\cdots,N,$$

In this paper, we concentrate on the reducible coupling matrix $\boldsymbol{A}$. Hence, we can assume that it has the following classic Perron-Frobenius normal form

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_{11} & 0 & \cdots & 0\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots\\ \boldsymbol{A}_{p1} & \boldsymbol{A}_{p2} & \cdots & \boldsymbol{A}_{pp}\end{bmatrix}$$

where $1\le p\le N$, $\boldsymbol{A}_{ii}\in R^{N_i\times N_i}\,(N_1+\cdots+N_p=N)$ is either a scalar or a square and irreducible sub-matrix, and for any $i>1$, there exists at least one j with $1\le j\le i-1$ such that $\boldsymbol{A}_{ij}\ne 0$. It is equivalent to that the digraph has a directed spanning tree. Without loss of generality, we only discuss the case $p=2$, that is $N_1+N_2=N$, and other cases can be dealt with similarly.

Corresponding to the Perron-Frobenius normal form of coupling matrix $\boldsymbol{A}$, we can change the system (2) into the following form

$$\begin{cases}\dot{x}_i(t)=f\left(x_i(t),t\right)+c\sum\limits_{j=1}^{N_1}a_{ij}\Gamma x_j(t),\ i=1,\cdots,N_1,\\ \dot{x}_i(t)=f\left(x_i(t),t\right)+c\sum\limits_{j=1}^{N}a_{ij}\Gamma x_j(t),\ i=N_1+1,\cdots,N,\end{cases}\qquad(3)$$

Then the synchronization of complex network model (3) is investigated. To realize the synchronization of network (3), some controllers will be added and the pinning controlled network can be described by

$$\begin{cases} \dot{x}_i(t) = f(x_i(t),t) + c\sum_{j=1}^{N_1} a_{ij}\Gamma x_j(t) - cd_i\Gamma(x_i - \overline{x}(t)), \\ \qquad i = 1,\cdots,l, \\ \dot{x}_i(t) = f(x_i(t),t) + c\sum_{j=1}^{N_1} a_{ij}\Gamma x_j(t),\ i = l+1,\cdots,N_1, \\ \dot{x}_i(t) = f(x_i(t),t) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) - cd_i\Gamma(x_i - \overline{x}(t)), \\ \qquad i = N_1+1,\cdots,N_1+m, \\ \dot{x}_i(t) = f(x_i(t),t) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t), \\ i = N_1+m+1,\cdots,N, \end{cases} \tag{4}$$

where $l < N_1$ and $m < N_2$.

Denote $\overline{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$, and the error vectors are defined as $\delta x_i = x_i - \overline{x}(t),\ i = 1,\cdots,N$.

Easily, we have that

$$\sum_{i=1}^{N}\delta x_i(t) = \sum_{i=1}^{N}(x_i(t) - \overline{x}(t)) = \sum_{i=1}^{N} x_i(t) - N\overline{x}(t) = 0 \tag{5}$$

Notice that $\sum_{j=1}^{N} a_{ij} = 0$, then the error system can be described by

$$\begin{cases} \dfrac{\mathrm{d}\delta x_i(t)}{\mathrm{d}t} = f(x_i(t),t) - f(\overline{x}(t)) + c\sum_{j=1}^{N_1}\tilde{a}_{ij}\Gamma\,\delta x_j(t) \\ \qquad + J,\ i = 1,\cdots,N_1, \\ \dfrac{\mathrm{d}\delta x_i(t)}{\mathrm{d}t} = f(x_i(t),t) - f(\overline{x}(t)) + c\sum_{j=1}^{N}\tilde{a}_{ij}\Gamma\,\delta x_j(t) \\ \qquad + J,\ i = N_1+1,\cdots,N, \end{cases} \tag{6}$$

where $d_i > 0$ and $\tilde{a}_{ii} = a_{ii} - d_i,\ i = 1,\cdots,l,N_1+1,\cdots,N_1+m$ $\tilde{a}_{ij} = a_{ij}$ otherwise, and

$$J = f(\overline{x}(t),t) - \frac{1}{N}\sum_{k=1}^{N}\left(f(x_k(t),t) + c\sum_{l=1}^{N} a_{kl}\Gamma x_l(t)\right).$$

Assumption 1. [9]There exists a constant matrix K such that

$$(x-y)^T\left(f(x,t) - f(y,t)\right) \le (x-y)^T K\Gamma(x-y)$$

for any $x, y \in R^n$, where $\Gamma > 0$.

Note that Assumption 1 is very mild. For example, all linear and piecewise linear functions satisfy Assumption 1. In addition, if $\partial f_i/\partial x_j$ $(i, j = 1,2,\cdots,n)$ is bounded and Γ is positive definite, Assumption 1 is also satisfied. Therefore, it holds for many well-known systems, such as the Lorenz system, Chen system, Lu system, recurrent neural networks, Chua's circuit, etc.

Lemma 1 [22]. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive definite Hermitian matrices. Then for each

$$k = 1,\cdots,n,\quad \begin{aligned} &\min\left[\lambda_1(\boldsymbol{A})\lambda_k(\boldsymbol{B}),\lambda_1(\boldsymbol{B})\lambda_k(\boldsymbol{A})\right] \ge \lambda_k(\boldsymbol{BA}) \\ &\ge \max\left[\lambda_n(\boldsymbol{A})\lambda_k(\boldsymbol{B}),\lambda_n(\boldsymbol{B})\lambda_k(\boldsymbol{A})\right]. \end{aligned}$$

Remark 1. Denote $\theta = \|K\|_1$. According to Lemma 1, we have the following inequality

$$(x-y)^T K\Gamma(x-y) \le \theta(x-y)^T\Gamma(x-y)$$

for any $x, y \in R^n$, which will be used repeatedly below.

Lemma 2 [11]. If $\boldsymbol{A} = (a_{ij})_{N\times N}$ is an irreducible matrix that satisfies $a_{ij} = a_{ji} \ge 0$ for $i \ne j$, and $\sum_{j=1}^{N} a_{ij} = 0$ for all $i = 1,2,\cdots,N$, then all eigenvalues of the matrix

$$\begin{pmatrix} a_{11}-d_1 & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22}-d_2 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN}-d_N \end{pmatrix}$$

are negative, where $d_1,\cdots,d_N$ are nonnegative constants and $d_1 + \cdots + d_N > 0$.

3. Main Results

In this section, some sufficient pinning criteria are derived to guarantee the globally asymptotical synchronization of complex dynamical networks.

In this subsection, we investigate the linearly coupled networks, where the coupling matrix is reducible and asymmetric.

Denote $\overline{X}(t) = \left(\overline{x}^T(t),\cdots,\overline{x}^T(t)\right)^T$,

$\tilde{A}_{11} = A_{11} - diag\{(1-1/N)d_1,\cdots,(1-1/N)d_l,0,\cdots,0\}$ and

$\tilde{A}_{22} = A_{22} - diag\{(1-1/N)d_{N_1+1},\cdots,$

$(1-1/N)d_{N_1+m},0\cdots,0\}$,

then we have the following theorem:

Theorem 1: Under the Assumption 1, if the following two inequalities

$$\begin{cases} 2\left(\theta I_{N_1} + c\tilde{A}_{11}^s\right)\otimes\Gamma + c\|A_{21}\|_\infty I_{N_1}\otimes\Gamma < 0 \\ 2\left(\theta I_{N_2} + cA_{22}^s\right)\otimes\Gamma + c\|A_{21}\|_1 I_{N_2}\otimes\Gamma < 0 \end{cases} \tag{8}$$

hold simultaneously, then the pinned network (4) can be globally asymptotically synchronized to $\overline{X}(t)$. **Proof**. Choose the Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N_1} \delta x_i^{\mathrm{T}}(t)\,\delta x_i(t) + \frac{1}{2}\sum_{i=N_1+1}^{N} \delta x_i^{\mathrm{T}}(t)\,\delta x_i(t)$$
$$= V_1(t) + V_2(t).$$

Denote

$\delta_1 = \left(\delta x_1^T(t) \;\cdots\; \delta x_{N_1}^{\mathrm{T}}(t)\right)^T$, $\delta_2 = \left(\delta x_{N_1+1}^{\mathrm{T}}(t) \;\cdots\; \delta x_N^{\mathrm{T}}(t)\right)^T$.

From (7), we have

$$\dot V_1(t)|_{(6)} = \sum_{i=1}^{N_1} \delta x_i^T(t)\Big(f(x_i(t),t) - f(\bar x(t),t)$$
$$+ c\sum_{j=1}^{N_1}\tilde a_{ij}\Gamma\delta x_j(t) + J\Big)$$
$$\le \sum_{i=1}^{N_1}\delta x_i^T(t) K\Gamma\, \delta x_i(t) + c\sum_{i=1}^{N_1}\delta x_i^T(t)\sum_{j=1}^{N}\tilde a_{ij}\Gamma\,\delta x_j(t)$$
$$+ \sum_{i=1}^{N_1}\delta x_i^T(t) J$$
$$= \delta_1^T\left(I_{N_1}\otimes K\Gamma\right)\delta_1 + c\,\delta_1^T\left(\tilde A_{11}^s\otimes\Gamma\right)\delta_1 + \sum_{i=1}^{N_1}\delta x_i^T(t) J$$
$$\le \delta_1^T\left[\left(\theta I_{N_1} + c\tilde A_{11}^s\right)\otimes\Gamma\right]\delta_1 + \sum_{i=1}^{N_1}\delta x_i^T(t) J$$

and

$$\dot V_2(t)|_{(6)} = \sum_{i=N_1+1}^{N} \delta x_i^T(t)\Big[f(x_i(t),t) - f(\bar x(t),t)$$
$$+ c\sum_{j=1}^{N}\tilde a_{ij}\Gamma\delta x_j(t) + J\Big]$$
$$= \sum_{i=N_1+1}^{N} \delta x_i^T(t)\Big[f(x_i(t),t) - f(\bar x(t),t)$$
$$+ c\sum_{j=N_1+1}^{N}\tilde a_{ij}\Gamma\delta x_j(t)\Big]$$
$$+ c\sum_{i=N_1+1}^{N}\delta x_i^T(t)\sum_{j=1}^{N_1}\tilde a_{ij}\Gamma\,\delta x_j(t) + \sum_{i=N_1+1}^{N}\delta x_i^T(t) J$$
$$\le \delta_2^T\left[\left(\theta I_{N_2} + c\tilde A_{22}^s\right)\otimes\Gamma\right]\delta_2 + \sum_{i=N_1+1}^{N}\delta x_i^T(t) J$$
$$+ c\sum_{i=N_1+1}^{N}\delta x_i^T(t)\sum_{j=1}^{N_1}a_{ij}\Gamma\,\delta x_j(t)$$

By Schwardz inequality, we can obtain

$$\delta x_i^T(t)\Gamma\,\delta x_j(t) \le 1/2\left(\delta x_i^T(t)\Gamma\,\delta x_i(t) + \delta x_j^T(t)\Gamma\,\delta x_j(t)\right)$$

Therefore,

$$c\sum_{i=N_1+1}^{N}\delta x_i^T(t)\sum_{j=1}^{N_1}a_{ij}\Gamma\,\delta x_j(t)$$
$$\le 1/2c\sum_{i=N_1+1}^{N}\sum_{j=1}^{N_1}a_{ij}\left(\delta x_i^T(t)\Gamma\,\delta x_i(t) + \delta x_j^T(t)\Gamma\,\delta x_j(t)\right)$$
$$\le 1/2c\|A_{21}\|_\infty\sum_{j=1}^{N_1}\delta x_j^T(t)\Gamma\,\delta x_j(t)$$
$$+ 1/2c\|A_{21}\|_1\sum_{i=N_1+1}^{N}\delta x_i^T(t)\Gamma\,\delta x_i(t)$$
$$= 1/2c\left[\delta_1^T\left(\|A_{21}\|_\infty I_{N_1}\otimes\Gamma\right)\delta_1\right.$$
$$\left. + \delta_2^T\left(\|A_{21}\|_1 I_{N_2}\otimes\Gamma\right)\delta_2\right]$$

According to (5), we can obtain the following equality

$$\sum_{i=1}^{N_1}\delta x_i^T(t)J + \sum_{i=N_1+1}^{N}\delta x_i^T(t)J = \left(\sum_{i=1}^{N}\delta x_i^T(t)\right)J = 0.$$

Combining with the above results, we have

$$\dot V(t) = \dot V_1(t) + \dot V_2(t)$$
$$\le \delta_1^T\left[\left(\theta I_{N_1} + c\tilde A_{11}^s\right)\otimes\Gamma\right]\delta_1 + \sum_{i=1}^{N_1}\delta x_i^T(t)J$$
$$+ \delta_2^T\left[\left(\theta I_{N_2} + c\tilde A_{22}^s\right)\right]\Gamma\,\delta_2$$
$$+ \frac{1}{2}c\left[\delta_1^T\left(\|A_{21}\|_\infty I_{N_1}\otimes\Gamma\right)\delta_1 + \sum_{i=N_1+1}^{N}\delta x_i^T(t)J\right.$$
$$\left. + \delta_2^T\left(\|A_{21}\|_1 I_{N_2}\otimes\Gamma\right)\delta_2\right]$$
$$= \frac{1}{2}\delta_1^T\left[2\left(\theta I_{N_1} + c\tilde A_{11}^s\right)\otimes\Gamma + c\|A_{21}\|_\infty I_{N_1}\otimes\Gamma\right]\delta_1$$
$$+ \frac{1}{2}\delta_2^T\left[2\left(\theta I_{N_2} + c\tilde A_{22}^s\right)\otimes\Gamma + c\|A_{21}\|_1 I_{N_2}\otimes\Gamma\right]\delta_2$$

Therefore, we can get $\dot V(t) < 0$ from (7). The proof is completed.

Obviously, the two conditions that $\tilde A_{11}^s$ and $\tilde A_{22}^s$ are negative are necessary for (8). Furthermore, we have to make the coupling strength c large enough, which meets the requirement of Theorem 1. Take $\Gamma = \gamma I_n$ and $\beta = \max\left\{\|A_{21}\|_\infty, \|A_{21}\|_1\right\}$, where $\gamma > 0$ is a constant, then we can obtain the following results which are more applicable.

Corollary 1: The pinned network (4) can be globally asymptotically synchronized to $\bar{X}(t)$, when the following two inequalities

$$\begin{cases} 2\theta + 2c\lambda_1(\tilde{A}_{11}^s) + c\beta < 0 \\ 2\theta + 2c\lambda_1(A_{22}^s) + c\beta < 0 \end{cases} \tag{9}$$

hold simultaneously.

The proof of Corollary 1 is obvious and is omitted here.

When the coupling strength c is large enough, if $\min\left\{\lambda_1\left(\tilde{A}_{11}^s\right), \lambda_1\left(\tilde{A}_{22}^s\right)\right\} < -\beta/2$ is satisfied, then it is easy for us to find that Corollary 1 holds. It happens that it provides us a direct way to compute that how many nodes should be pinned. Subsequently, we can obtain that the coupling strength c must be larger than $\max\left\{-2\theta\Big/\left(2\lambda_1\left(\tilde{A}_{11}^s\right)+\beta\right), -2\theta\Big/\left(2\lambda_1\left(\tilde{A}_{22}^s\right)+\beta\right)\right\}$

in order to reach synchronization. These results would be useful in pinning scheme by numerical simulations. Naturally, we can get the following corollary.

Corollary 2: Suppose that Assumption 1 holds and Γ is positive definite. The pinned network (4) is globally synchronized if the coupling strength c satisfies the following inequality

$$c > \max\left\{-2\theta\Big/\left(2\lambda_1\left(\tilde{A}_{11}^s\right)+\beta\right), -2\theta\Big/\left(2\lambda_1\left(\tilde{A}_{22}^s\right)+\beta\right)\right\}. \tag{10}$$

Denote

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - diag\left\{\left(1-\frac{1}{N}\right)d_1, \cdots, \left(1-\frac{1}{N}\right)d_l, 0, \cdots, 0\right\}$$

If the modified coupling matrix $\tilde{A}$ is irreducible, then it is easy to get the following corollary.

Corollary 3: Suppose Assumption 1 holds. The controlled network (4) can be globally synchronized to $\bar{X}(t)$, provided the condition $\theta + c\lambda_1\left(\tilde{A}^s\right) < 0$ is satisfied.

Proof. Take the Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} x_i^T(t)\, x_i(t)$$

When the condition $\theta + c\lambda_1\left(\tilde{A}^s\right) < 0$ holds, by similar methods in Theorem 1, we can gain that $\bar{X}(t)$ is globally asymptotically stable.

Remark 2: If Corollary 1 holds, then $\lambda_1\left(\tilde{A}^s\right)$ must be negative. Consequently, the synchronization of the controlled network can be realized as long as the coupling strength $c > \theta\Big/\left(-\lambda_1\left(\tilde{A}^s\right)\right)$. Clearly, the smaller $-\lambda_1\left(\tilde{A}^s\right)$ is, the larger the coupling strength has to be, which means we need spend more cost on achieving the synchronization of the pinned network. Then, it is nature to make $-\lambda_1\left(\tilde{A}^s\right)$ larger. By the Courant-Fischer Theorem, we can obtain the conclusion that $-\lambda_1\left(\tilde{A}^s\right)$ increases as the feedback control gains $d_1, \cdots, d_l$ increase.

We conclude that the sufficient condition in Corollary 1 and the one in Corollary 3 could be compared formally. For the complex network which has a directed spanning tree, we must spend more cost on driving it to synchronize. And, it is coincident that this extra cost spent in the reducible network is caused by A_{21}. It is reasonable and practical just because its corresponding strongly connected network possesses more links, such as the internet, the WWW net, which guarantees its synchronization can be reached with fewer external energy.

It is not difficult to find that the theoretical coupling strength given in (10) is too conservative. Usually, it is much larger than the needed value. Obviously, the intuitive idea is to make the coupling strength as small as possible. Here, the adaptive technique[23-24] is adopted to achieve this goal. The selected pinning controllers in (4), associated with the adaptive coupling law, lead to

$$\begin{cases} \dot{x}_i(t) = f\left(x_i(t), t\right) + c(t)\sum_{j=1}^{N_1} a_{ij}\Gamma x_j(t) \\ \quad -c(t)d_i\Gamma\left(x_i - \bar{x}(t)\right),\ i = 1, \cdots, N_1, \\ \dot{x}_i(t) = f\left(x_i(t), t\right) + c(t)\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) \\ -c(t)d_i\Gamma\left(x_i - \bar{x}(t)\right), i = N_1 + 1, \cdots, N, \end{cases} \tag{11}$$

with

$$\dot{c}(t) = \begin{cases} c_1(t) = \mu_1\sum_{j=1}^{N_1}\left(x_j - \bar{x}(t)\right)\Gamma\left(x_j - \bar{x}(t)\right), \\ \quad i = 1, 2, \cdots, N_1, \\ c_2(t) = \mu_2\sum_{j=N_1+1}^{N}\left(x_j - \bar{x}(t)\right)\Gamma\left(x_j - \bar{x}(t)\right), \\ \quad i = N_1 + 1, \cdots, N, \end{cases} \tag{12}$$

where μ_1 and μ_2 are small positive constants, and $d_i > 0, i = 1, \cdots, l, N_1 + 1, \cdots, N_1 + m$ $\left(l < N_1, m < N_2\right)$ and $d_i = 0$ otherwise.
Define error vector as

$$\delta x_i = x_i - \bar{x}(t), i = 1, \cdots, N$$

and notice that $\sum_{j=1}^{N} a_{ij} = 0$, then the error system of (11) can be described by

$$\begin{cases} \dfrac{d\delta x_i(t)}{dt} = f(x_i(t),t) - f(\overline{x}(t),t) + J \\ + c(t)\sum_{j=1}^{N}\tilde{a}_{ij}\Gamma\delta x_j(t),\ i = 1,\cdots,N_1, \\ \dfrac{d\delta x_i(t)}{dt} = f(x_i(t),t) - f(\overline{x}(t),t) + J \\ + c(t)\sum_{j=1}^{N}\tilde{a}_{ij}\Gamma\delta x_j(t),\ i = N_1+1,\cdots,N \end{cases} \quad (13)$$

with

$$c(t) = \begin{cases} \dot{c}_1(t) = \mu_1 \sum_{j=1}^{N_1} \delta x_j^T(t)\Gamma\delta x_j(t),\ i = 1,\cdots,N_1 \\ \dot{c}_2(t) = \mu_2 \sum_{j=N_1+1}^{N} \delta x_j^T(t)\Gamma\delta x_j(t) \\ i = N_1+1,\cdots,N, \end{cases} \quad (14)$$

where $\tilde{a}_{ii} = a_{ii} - d_i,\ i = 1,\cdots,l, N_1+1,\cdots,N_1+m$, and $\tilde{a}_{ij} = a_{ij}$ otherwise, and

$$J = f(\overline{x}(t),t) - \frac{1}{N}\sum_{k=1}^{N}\left(f(x_k(t),t) + c(t)\sum_{l=1}^{N} a_{kl}\Gamma x_l(t)\right).$$

Theorem 2. Under Assumption 1, the adaptively controlled weakly connected network (11) with reducible and asymmetric coupling matrix can be globally synchronized for two small constants $\mu > 0$ and $\mu > 0$.

Proof. Consider the Lyapunov functional as follows:
$V(t) = V(t) + V(t)$,
Where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N_1} \delta x_i^T(t)\delta x_i(t) + \frac{\sigma_1}{2\mu_1}\left(c(t)-\nu_1\right)^2$$

$$V_2(t) = \frac{1}{2}\sum_{i=N_1+1}^{N} \delta x_i^T(t)\delta x_i(t) + \frac{\sigma_2}{2\mu_2}\left(c(t)-\nu_2\right)^2$$

and $\sigma_1, \sigma_2, \nu_1, \nu_2$ are positive constants to be determined below.

The derivative of $V_1(t)$ along the trajectories of (13) gives

$$\dot{V}_1(t)\Big|_{(13)} = \sum_{i=1}^{N_1} \delta x_i^T(t)\Big(f(x_i) - f(\overline{x}) + \sigma_1(c(t)-\nu_1) + c(t)\sum_{j=1}^{N}\tilde{a}_{ij}\Gamma\delta x_j(t) + J\Big)$$

$$\le \sum_{i=1}^{N_1} \delta x_i^T(t)K\Gamma\delta x_i(t) + c_1(t)\delta_1^T\left(\tilde{A}_{11}^s \otimes \Gamma\right)\delta_1$$

$$+ \sum_{i=1}^{N_1} \delta x_i^T(t)J + \sigma_1(c(t)-\nu_1)\delta_1^T\left(I_{N_1}\otimes\Gamma\right)\delta_1$$

$$\le \delta_1^T\Big(\theta I_{N_1} + c(t)\tilde{A}_{11}^s + \sum_{i=1}^{N_1} \delta x_i^T(t)J + \sigma_1(c(t)-\nu_1)I_{N_1}\Big)\otimes\Gamma\delta_1.$$

Similarly, the differential coefficient of $V_2(t)$ is described by

$$\dot{V}_2(t)\Big|_{(13)} \le \delta_2^T\left(\theta I_{N_2} + c(t)\tilde{A}_{22}^s + \sigma_2(c(t)-\nu_2)I_{N_2}\right)\otimes\Gamma\delta_2 + \sum_{i=N_1+1}^{N}\delta x_i^T(t)J + c(t)\sum_{N_1+1}^{N}\sum_{j=1}^{N_1} a_{ij}\delta x_i^T(t)\Gamma\delta x_j(t)$$

However, by some basic inequalities, we have

$$c(t)\sum_{N_1+1}^{N}\sum_{j=1}^{N_1} a_{ij}\delta x_i^T(t)\Gamma\delta x_j(t)$$

$$\le \frac{1}{2}c(t)\sum_{N_1+1}^{N}\sum_{j=1}^{N_1} a_{ij}\left(\delta x_i^T(t)\Gamma\delta x_i(t) + \delta x_j^T(t)\Gamma\delta x_j(t)\right)$$

$$\le \frac{1}{2}c(t)\left(\|A_{21}\|_\infty \delta_1^T\left(I_{N_1}\otimes\Gamma\right)\delta_1 + \|A_{21}\|_1 \delta_2^T\left(I_{N_2}\otimes\Gamma\right)\delta_2\right)$$

Therefore, we can obtain that

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t)$$

$$\le \delta_1^T\left[\left(\theta + \sigma_1 c(t) - \sigma_1\nu_1 + \frac{1}{2}\|A_{21}\|_\infty c(t)\right)I_{N_1} + c(t)\tilde{A}_{11}^s\right]\otimes\Gamma\delta_1$$

$$+ \delta_2^T\left[\left(\theta + \sigma_2 c(t) - \sigma_2\nu_2 + \frac{1}{2}\|A_{21}\|_1 c(t)\right)I_{N_2} + c(t)\tilde{A}_{22}^s\right]\otimes\Gamma\delta_2$$

From Lemma 2, one can see that $\sigma_1 I_{N_1} + \tilde{A}_{11}^s$ and $\sigma_2 I_{N_2} + \tilde{A}_{22}^s$ are negative definite when σ_1 and σ_2 are sufficiently small. If we let the initial conditions $c(0)$ be large and μ_1 and μ_2 be small enough, then $c(t)$ increase very slowly. In this case, if c_1 are given largely enough, then the synchronization can be achieved by Corollary 2. Once the synchronization is achieved, $c(t)$ converge to two constants and is bounded. Since $c(t)$ are bounded, we can always choose two sufficient large constants ν_1 and ν_2 such that

$$\left(\theta + \sigma_1 c(t) - \sigma_1\nu_1 + \frac{1}{2}\beta c(t)\right)I_{N_1} + c(t)\tilde{A}_{11}^s < 0$$

and

$$\left(\theta+\sigma_2 c(t)-\sigma_2 v_2+\frac{1}{2}\beta c(t)\right)I_{N_2}+c(t)\tilde{A}_{22}^s<0$$

hold simultaneously, where $\beta=\max\left\{\|A_{21}\|_\infty,\|A_{21}\|_1\right\}$. Thus, $\dot{V}$ is always negative as time evolves. Hence, the pinned synchronization can be realized. And $c_1(t)$ and $c_2(t)$ tend to constants respectively. Thus, the proof of Theorem 2 is completed.

Remark 3: Notice that the synchronization conditions in Theorem 2 are independent of the network structure and the number of pinned nodes. Therefore, according to (12) by Theorem 2, under any fixed network structure and pinning controllers in the form of (6), the coupling strengths can be self-adaptively determined to achieve network synchronization.

4. Numerical Simulations

In this section, two simulation examples are given to verify the conclusions established above.

4.1 Validity of Theorem 1

In this numerical simulation, we consider a small-world network with 100 nodes. The connection weights are chosen randomly in [0,10] with uniform distribution. Here, we consider network (4) that consists of N identical Chen systems, where $\Gamma=diag\{1,2,1\}$, $\theta=30.9342$ [7] and

$$f(x_i,t)=\begin{cases}35(x_{i2}-x_{i1}),\\-7x_{i1}-x_{i1}x_{i3}+28x_{i2},\\x_{i1}x_{i2}-3x_{i3}.\end{cases}$$

Define $E(t)=\sqrt{\sum_{i=1}^{100}\|x_i(t)-\overline{x}(t)\|^2/100}$, which is used to measure the synchronization quality. Because the coupling matrices are produced randomly by matlab program, it needs many pinned nodes for attaining negative $\lambda_1(\tilde{A}_{11}^s)$ and $\lambda_1(\tilde{A}_{22}^s)$ simultaneously. Hence, we choose five coupling matrices $A_i, i=1,\cdots,5$ at random firstly. Then, take

$$\boldsymbol{A}=\frac{1}{5}\sum_{i=1}^{5}A_i.$$

Let $d_1=\cdots=d_l=d_{N-m+1}=\cdots=d_N=4$. Thus, we get $\beta=9.87$. If the combination of l and m make the inequality $\min\left\{\lambda_1(\tilde{A}_{11}^s),\lambda_1(\tilde{A}_{22}^s)\right\}<-\beta/2$ hold, then we say that the combination is valid. Moreover, we rule that $c_{min}<d=40$ is acceptable. **Table 1** reflects the valid relationships among l, m, $\lambda_1(\tilde{A}_{11}^s)$, $\lambda_1(\tilde{A}_{22}^s)$ and the least threshold c_{min} of coupling strength.

Table 1. Valid relationships among l, m, $\lambda_1(\tilde{A}_{11}^s)$, $\lambda_1(\tilde{A}_{22}^s)$ and the least threshold c_{min} of coupling strength.

l	m	$\lambda_1(\tilde{A}_{11}^s)$	$\lambda_1(\tilde{A}_{22}^s)$	c_{min}
2	1	−4.9940	−5.4335	524.3085
3	2	−5.1388	−5.7295	151.7870
4	2	−5.3221	−5.7295	79.9127
5	2	−5.5792	−5.7295	48.0196
6	2	−5.7514	−5.7295	38.9354

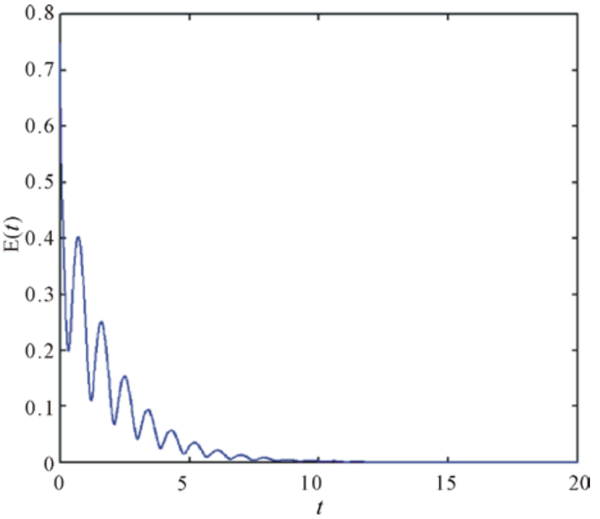

Figure 1. The graph displays the synchronization process of the system (4).

According to the above two rules, $l=6$ and $m=2$ can meet our needs. Choose any initial values for x_i from $[0,1]$. Take $c=39$ and $d=40$, then we can get the following synchronization graph **Figure 1**.

4.2. Verifying the Effectiveness of Theorem 2

In this simulation, all involved parameters are the same as those in Section 4.1 and the uncoupled chaotic system is still Chen system. And, we still use $E(t)=\sqrt{\sum_{i=1}^{100}\|x_i(t)-\overline{x}(t)\|^2/100}$ to reflect the synchronization quality. Then, we can obtain the following graph **Figure 2**, where the top one shows the synchronization process of the system (8) and the two ones in bottom display the limit process of $c_1(t)$ and $c_2(t)$.

5. Conclusions

This paper considered the globally synchronization for a class of linearly coupled complex networks with reducible and asymmetrical coupling configuration. On

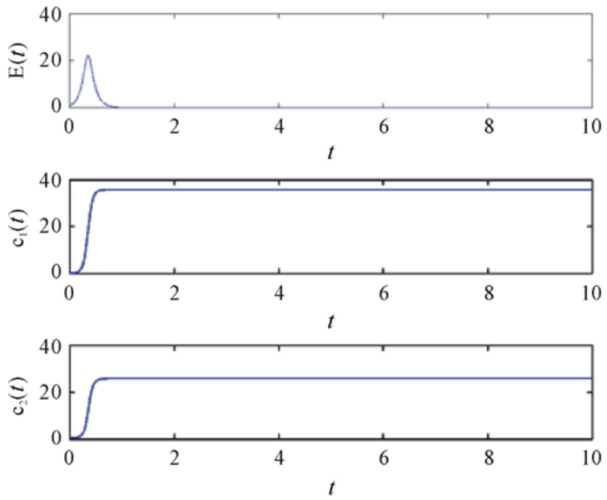

Figure 2. The graph displays the synchronization process of the system (10) and the limit process of $c_1(t)$ and $c_2(t)$.

the basis of Assumption 1 and some lemmas, we obtained some theorems and corollaries by Lyapunov direct method which ensured the synchronization of pinned complex networks could be realized indeed. Moreover, by further analysis on the synchronization conditions in Theorem 1, one could obtain a detailed method for solving the two key problems in pinning synchronization: how many nodes should be pinned and how large coupling strength should be applied to realize the synchronization of pinned complex network. Moreover, it also revealed the relationships between the differences of two convergence conditions and the Perron- Frobenius normal form of coupling matrix. At last, we provided two numerical simulations to verify the validity of those results.

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