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Quantum Features of Atom–Field Systems in the Framework of Deformed Fields

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Abstract: We propose a new kind of Schrödinger cat state introduced as a superposition of spin coherent states in the framework of noncommutative spaces. We analyze the nonclassical features for these noncommutative deformed states in terms of the main physical parameters. The physical importance of deformed states is that they provide a convenient description of a large set of laser systems. As an application, we develop the Jaynes–Cummings model by considering the interaction among atoms and cat state fields associated to deformed spin algebras. In this context, we show the dynamical behavior of the nonlocal correlation and nonclassical properties in these quantum systems.

Keywords: deformed coherent states; cat deformed states; noncommutative spaces; deformed su(2) algebra; real and ideal lasers; photon antibunching; sub-Poissonian photon statistics; entanglement

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1. Introduction

In almost all the areas of physics, the theory of coherent states (CSs) has been used after Schrödinger first built quantum states which are the closest to replicate the classical behavior [1]. Glauber was first to observe the notion of coherence linked to these physical states [2,3]. Similar states were also reinstated by Klauder [4,5], with a widely accepted view that the CSs are linked to the harmonic oscillators (HOs). Due to their vital characteristics, these CSs were simplified to other quantum systems from either a mathematical or physical perspective [6–8]. Gilmore [9] and Perelomov [10] separately presented CSs linked to any set, not only to the Heisenberg group for an HO. An example of an occurrence of these CSs is the spin coherent states (SCSs) connect with su(2) group. These CSs explain various systems and have several usages in statistical mechanics, quantum optics, and physics of condensed matter [6–8,11–13]. Moreover, the quantum groups have been used as a mathematical explanation of deformed Lie algebras [14] that made the construction of the deformed coherent states (DCSs) possible. The DCSs are presented as the actual expansion of the concept of CSs. A general deformation of the Glauber CSs was created [15] as connected to deformed quantum HOs. Similarly, deformation of the SCSs is created as states connected to the deformed spin algebra $su_q(2)$ [16,17]. Over the last decades, these DCSs have attracted a great attention due to their potential applications in diverse branches of the physics [18–21]. It has been shown experimentally that the real lasers, bunched, and antibunched lights provide photon distributions that can be sub-Poissonian or super-Poissonian [22,23]. On the other hand, there are interesting quantitative effects, and related states that are difficult to be prepared and detected, namely superposition quantum states displaying interference effects [24]. Such states exhibit the surprising results of the superposition principle. Furthermore, superposed and nonlinear CSs can be generated within the motion of trapped ions [25]. The lights with nonclassicality properties are actually a popular area of research, and scientists are interested in further revealing the fundamental truth of the quantum world.

In the Jaynes–Cummings model (JCM) [26], the interaction involving radiation and matter is the most effective and simplest system in quantum optics which explains the interaction of two-level atoms with an electromagnetic field. Mathematically, JCM is well described and can as well be resolved precisely [27]. Furthermore, a recent technological breakthrough that made the production of high-Q cavities possible with a possibility to scientifically understand this relatively ideal system for a Rydberg atom in a superconductor cavity [28]. Therefore, much interest has been given to the investigation of JCM and several fascinating quantum occurrences have been speculated: For example Rabi oscillations [29], revival–collapse phenomena of atomic population inversion [30], photon anti-bunching [31], squeezing of information entropy [32], and the atomic dipoles [33].

Quantum entanglement is the most important aspects that differentiate quantum mechanics from the classical correlative. It is a very important aspect that carries out a significant function in different areas of the quantum information theory, for example, quantum key distribution [34–38], quantum computing [39] and teleportation [40–42]. Hence, the classification and the quantification of the quantum entanglement have gained ample interest and became highly researched areas in the last few decades. To measure the entanglement, a series of quantifiers have been suggested, for example, concurrence [43,44], entanglement of formation [45,46] and linear entropy [47,48]. The essential question for the quantum entanglement is to analyze if a particular state of the whole system, comprising of more than one subsystem, is entangled or not and it is factorizable if it can be given as a product of the subsystem states.

The aim of this manuscript is to introduce a new kind of Schrödinger cat states using the $\mathrm{su}_q(2)$ algebra. We investigate the nonclassical properties of these DCSs with respect to the main physical parameters, and we consider the JCM to describe the interaction among two-level atoms and cat state fields associated to $\mathrm{su}_q(2)$ algebra. Finally, we study the dynamical behavior of quantifiers of nonclassicality and nonlocal correlation for the considered bipartite quantum system.

The paper is organized as follows. In Section 2 we describe the deformed spin algebra and its properties. In Section 3, we introduce the SCSs and new kind of cat states in the framework of deformed spin algebra. In Section 4, the physical features of these DCSs are examined through the Mandel's parameter. Moreover, we study the interaction of a two-level atom with a field initially defined in deformed spin coherent states (DSCSs). The conclusion of this paper is given in the last section.

2. Deformed Spin Algebra

Quantized universal enveloping algebras refer to some specific deformations of Lie algebras. A considerable example is the case of deformed spin algebra (DSA), which was firstly introduced by Sklyanin [49]. Many interesting applications were developed in different areas of the physics using the bosonic realization in terms of the deformed operators, and the DSA has attracted great interest in the framework of mathematical and physical problems [50]. The generators $J^q_\pm$ and J^q_z are used to generate the $\mathbf{su}_q(2)$ algebra with the following commutation relations,

$$[J^q_z, J^q_\pm] = \pm J^q_\pm; \;\; [J^q_+, J^q_-] = [2J^q_z]_q, \tag{1}$$

where the function "[]" expresses the deformation of the algebra. In fact, by a specific choice of this function we have a particular deformation of the spin algebra. In the case $[y]_q = y$ we get the non-deformed algebra, and we consider the known deformation of the spin algebra for the following function [13,14]

$$[y]_q = \frac{1-q^y}{1-q}. \tag{2}$$

A significant property in the deformation version can be obtained for a specific values of the parameter q. For $q = 1$ we recover the non-deformed algebra.

The generators $J^q_\pm$ and J^q_z act on the orthonormal basis of the representation space $|j, m\rangle$ as

$$\begin{aligned} J^q_z|j,\, m\rangle &= m\,|j,\, m\rangle\,, \\ J^q_\pm|j,\, m\rangle &= \Big([j \mp m]_q [j \pm m + 1]_q\Big)^{\frac{1}{2}} |j,\, m \pm 1\rangle \end{aligned} \tag{3}$$

with $m = j, \ldots, -j$.

As commonly reported, it is possible to map the $\mathrm{su}_q(2)$ operators to HO operators by using the Jordan-Schwinger map [51,52] and Holstein–Primakoff realization (HPR) [53]. These realizations can be generalized, making it possible to achieve the operators of the deformed $\mathbf{su}_q(2)$ algebra by annihilation and creation operators of the deformed Hos. This is a significant result as it permits the deformation of the HOs to relate with the deformation of $\mathbf{su}(2)$ algebra [54]. For this study, the deformation of the HPR will be applied as:

$$J^q_+ = a^\dagger_q \sqrt{[2j - N]_q},\; J^q_- = \sqrt{[2j - N]_q}\, a_q,\; J^q_z = N - j, \tag{4}$$

where $a^\dagger_q$, a_q denote the deformed creation and annihilation operators. These operators act on the Fock states $|n\rangle$ as

$$a^+_q|n\rangle = \sqrt{[n+1]_q}|n+1\rangle,\, a_q|n\rangle = \sqrt{[n]_q}|n-1\rangle,\, N|n\rangle = \sqrt{[n]_q}|n\rangle, \tag{5}$$

and obeying the following commutation relations

$$a_q a^+_q - q a^+_q a_q = 1,\; \Big[N,\, a^+_q\Big] = a^+_q,\; \big[N,\, a_q\big] = -a_q,\, \text{for}\, 0 < q \le 1. \tag{6}$$

The states $|n\rangle,\ n = 0, 1, \ldots,$ comprise the complete orthonormal basis of the irreducible representation space.

3. Deformed Schrödinger Cat Spin Coherent States

A SCS is obtained through acting a displacement operator on the slightest weight state of the space basis [7,9]. Nevertheless, for a deformed algebra, it is not insignificant to create such an operator as the framework of the algebra is not conserved. The DSCSs are introduced as

$$|z,\, j\rangle_q = \mathbf{N}_q\Big(|z|^2\Big)\, E_q^{zJ^q_+}|j, -j\rangle\,,\;\; z \in \mathbf{C}\,, \tag{7}$$

where, we considered the deformed exponential

$$E^x_q = \sum_{m=0}^{\infty} \frac{x^m}{[m]_q!}\,,\quad [m]_q! = [m]_q[m-1]_q \ldots [1]_q,\;\; [0]! = 1. \tag{8}$$

The normalization function is given by

$$\mathbf{N}_q\Big(|z|^2\Big) = \frac{1}{\sqrt{(1 + |z|^2)^{2j}_q}}, \tag{9}$$

where, the Newton's deformed binomial formula is used

$$(x+y)^n_q \;:= \sum_{m=0}^{n} \left[{n \atop m}\right]_q x^{n-m} y^m. \tag{10}$$

Here, the deformed binomial function is,

$$\left[{n \atop m}\right]_q = \frac{[n]_q!}{[n]_q![n-m]_q!} \text{ for } n \geq m. \tag{11}$$

Using these formulations, the generalized spin coherent is written as:

$$|z,\, j\rangle_q = \left((1+|z|^2)_q^{2j}\right)^{-\frac{1}{2}} \sum_{m=-j}^{j} \left(\left[{2j \atop j+m}\right]_q\right)^{\frac{1}{2}} z^{(m+j)}|j,\, m\rangle. \tag{12}$$

To examine the physical features of the deformed states, we display the basis states $|j,\, m\rangle$ as a function of the Fock states $|n\rangle$ ($|j,\, m\rangle \sim |n\rangle$). To do this, we use the deformation of HPR given in Equation (4). Applying this realization, we obtain the change of variable $n = j + m$, which When used in Equation (11), gives the following deformed states

$$|z,\, j\rangle_q = \left((1+|z|^2)_q^{2j}\right)^{-\frac{1}{2}} \sum_{n=0}^{2j} \left(\left[{2j \atop n}\right]_q\right)^{\frac{1}{2}} z^n|n\rangle. \tag{13}$$

The Klauder's criteria for the DSCSs are widely studied in [55]. The realization of the $\mathrm{su}_q(2)$ quantum algebra with respect to the deformed HO operators has originated loads of studies in different areas. The DSCSs have been employed to explain a sizeable group of quantum systems made from different potentials, for example, Morse, Pöschl-Teller and infinite potentials.

Let us now introduce the deformed Schrödinger cat spin states and study their statistical properties of photons. Here, we consider superposition of two DSCSs, which are often called $\mathbf{su}(2)$ Schrödinger cat states, and discuss the physical properties of the photon statistics. We define $\mathbf{su}(2)$ Schrödinger cat states as

$$|z,\, j,\, \Phi\rangle_q = \mathbf{N}\left(|z|^2,\, j,\, \Phi\right) \left[|z,\, j\rangle_q + e^{i\Phi}|-z,\, j\rangle_q\right], \tag{14}$$

where $0 \leq \Phi \leq 2\pi$ is an adjustable angle and the normalization constant $\mathbf{N}_q\left(|z|^2,\, j,\, \Phi\right)$ is obtained from the normalization condition ${}_q\langle z,\, j,\, \Phi|z,\, j,\, \Phi\rangle_q = 1$,

$$\mathbf{N}_q\left(|z|^2,\, j,\, \Phi\right) = \frac{1}{\sqrt{2}}\left[1 + \frac{(1-|z|^2)_q^{2j}}{(1+|z|^2)_q^{2j}} \cos\Phi\right]^{-\frac{1}{2}}. \tag{15}$$

The deformed cat spin states exhibit a richer structure than the non-DSCSs. We examine the physical features of these states by evaluating the Mandel's parameter [56]. This parameter is given by

$$M_p = \frac{\langle(\Delta N)^2\rangle - \langle N\rangle}{\langle N\rangle}, \tag{16}$$

where $\langle N\rangle$ is the mean photon number and $\langle(\Delta N)^2\rangle$ is the photon number variance. This parameter depends on z, j, q and Φ. Let's start with the non-deformed case that corresponds to the CSs defined by Equation (13). Figure 1, displays the graph of Mandel's M_p parameter in terms of $|z|$ for specific values of the parameters q and j. We can see from the first graph that $M_p < 0$ for various values of the physical parameters. Interestingly, the parameter M_p decreases with $|z|$ and approaches the value -1 as $|z|$ becomes significantly large. This shows that the DSCSs are always sub-Poissonian and they are closest to classical states (photon states with Poissonian distribution) in the case of $q = 1$ with small values of j.

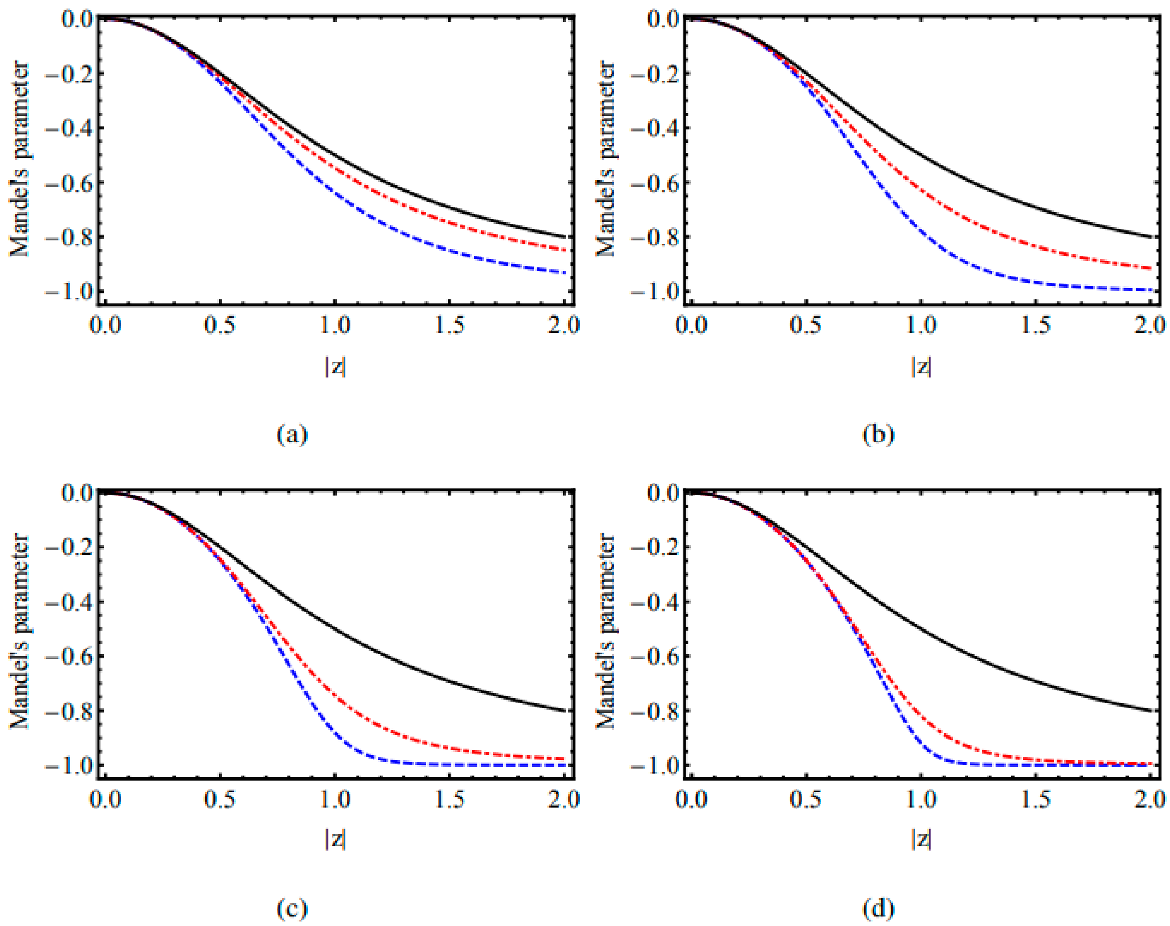

Figure 1. The parameter M_p of the spin coherent states (SCSs) in the context of $\mathrm{su}_q(2)$ algebra, given by Equation (11), as function of $|z|$ for specific values of the deformed parameter q and j. (**a**) is for $j = 1$, (**b**) is for $j = 2$, (**c**) is for $j = 4$, and (**d**) is for $j = 6$. Black (solid line) corresponds to $q = 1$ which displays the case of SCSs, blue (dashed line) corresponds to $q = 0.3$, and red (dotted-dashed line) corresponds to $q = 0.8$. The values of the parameter M_p are always negative in terms of $|z|$, q and j, displaying a sub-Poissonian distribution of photons.

In what follows, we consider the deformed Schrödinger cat states corresponding to Equation (14) and show their statistical properties. Figure 2 displays the variation of the Mandel's M_P parameter as a function of $|z|$ for various values of q and Φ with $j = 2$. The Mandel's parameter is significantly affected by the value of the phase Φ. We see from the Figure 2a,c, corresponding to the cases $\Phi = 0$ and $\Phi = \pi/4$, the deformed Schrödinger cat states exhibit super-Poissonian and sub-Poissonian distributions. We mention an interesting property of the states, where the Mandel's parameter depends on the values of the deformed parameter q at $|z| = 0$. For the other cases where $\Phi = \pi/2$ and $\Phi = \pi$, as shown in Figure 2b,d, the Mandel's parameter of the deformed cat states satisfies $M_p < 0$ providing the same distribution of the photons as in the case of the DSCSs.

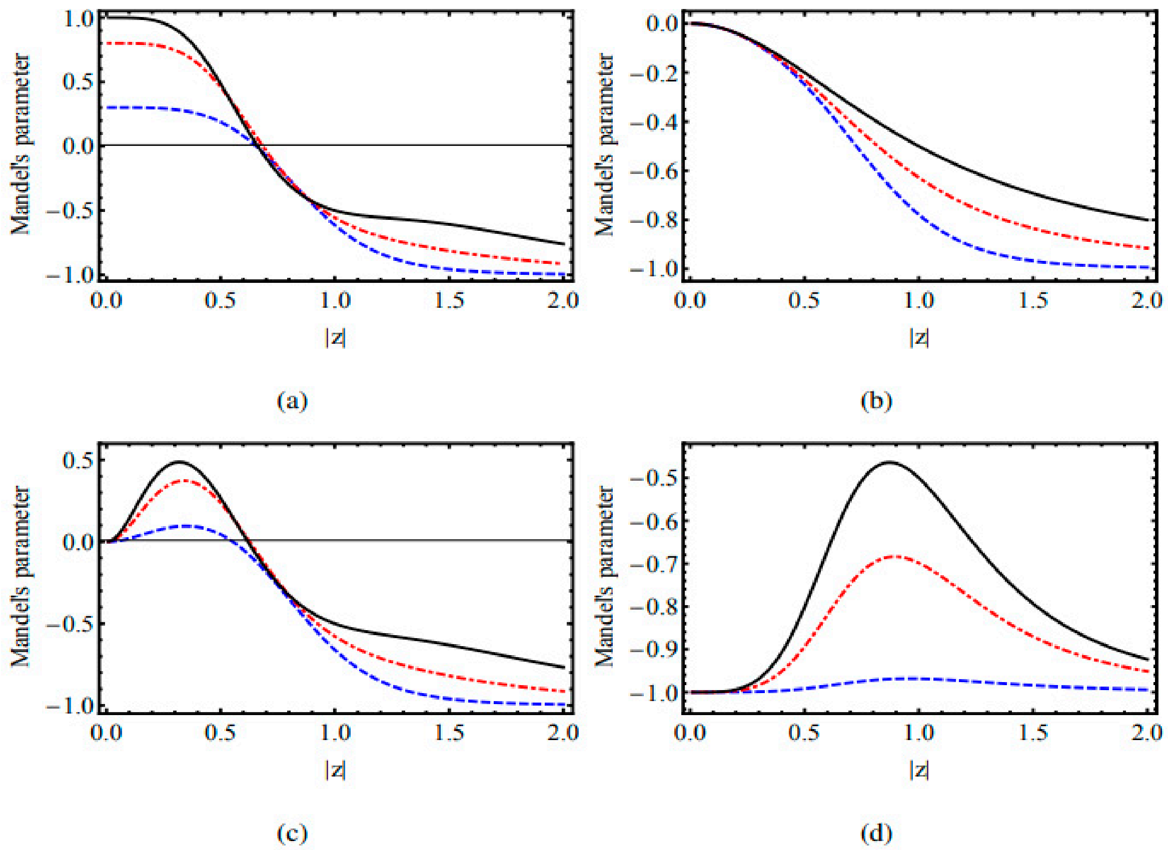

Figure 2. The parameter M_p of the SCSs in the context of su$_q$(2) algebra, defined in Equation (13), in terms of $|z|$ for specific values of the deformed parameter q with $j = 2$. (**a**) is for $\Phi = 0$, (**b**) is for $\Phi = \pi/2$, (**c**) is for $\Phi = \pi/4$, and (**d**) is for $\Phi = \pi$. Black (solid line) corresponds to $q = 1$ which displays the case of ordinary SCSs, blue (dashed line) corresponds to $q = 0.3$, and red (dotted-dashed line) corresponds to $q = 0.8$. The parameter M_p is negative or positive depending of values of the physical parameters, exhibiting sub-Poissonian and super-Poissonian distribution of photons.

4. Interactions of a Two-Level Atom with a Field in the Framework of su$_2$(2) Algebra

We consider an atomic system relating to a single field state described as the superposition of DSCSs. Let $|e\rangle$ and $|g\rangle$ represent the upper state and lower state of the atom. The interaction Hamiltonian is expressed as

$$\hat{H}_I = \hbar\mu\left(J_-^q|e\rangle\langle g| + J_+^q|g\rangle\langle e|\right), \tag{17}$$

where μ represents the atom–field coupling constant and J_-^q (J_+^q) is the lowering (raising) operator in the quantized field.

We assume that the two-level atom is firstly made in the upper state and the quantized field in the deformed cat spin coherent states (DCSCSs) in the context of HPR. The principle of linear superposition is the core of quantum mechanics via the control of the features of the single states that make them less or more pronounced. Considering the initial state of the bipartite system as a product state of quantum subsystems $|\psi(0)\rangle_{AF} = |\psi(0)\rangle_A \otimes |\psi(0)\rangle_F$. The wave function of the atom–field system is expended with respect to the states $|g, n+1\rangle$ and $|e, n\rangle$. The atom–field state at subsequent time t is given as

$$|\psi(t)\rangle_{AF} = \sum_{n=0}^{\infty}\left[\mathbf{A}_{e,n}|e, n\rangle + \mathbf{A}_{g,n+1}|e, n+1\rangle\right], \tag{18}$$

where the coefficients $\mathbf{A}_{g,n+1}(t)$ and $\mathbf{A}_{e,n}(t)$ are the probability amplitudes of the atom, which satisfy the two coupled differential equations obtained from the Schrödinger equation

$$\mathbf{i}\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{A}_{e,n}(t) = g\sqrt{[2j-N]_q}\sqrt{[n+1]_q}\mathbf{A}_{g,n+1}(t)$$

$$\mathbf{i}\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{A}_{g,n+1}(t) = g\sqrt{[2j-N]_q}\sqrt{[n+1]_q}\mathbf{A}_{e,n}(t)$$

The values of $\mathbf{A}_{g,n+1}(0)$ and $\mathbf{A}_{e,n}(0)$ are obtained from the preliminary requirements. According to the two-level atom dynamics, the probabilities are significant when the two-level atom is in the excited state, and these possibilities are defined by $|\mathbf{A}_{g,n+1}|^2$ and $|\mathbf{A}_{e,n}|^2$ for $t > 0$. On this point, a measurable inversion population W was described as the difference on the probability when the atomic system is in the ground and excited state. Since the Hamiltonian operator of the atom is proportional to the operator $|g\rangle\langle g| - |e\rangle\langle e|$, the population inversion is precisely the atomic energy and it is expressed as

$$\begin{aligned} W &= -\mathbf{Tr}[\sigma_z \rho_{\mathbf{A}}] \\ &= \sum_{n=0}^{\infty}\left(|\mathbf{A}_{e,n}|^2 - |\mathbf{A}_{g,n+1}|^2\right). \end{aligned} \tag{19}$$

To quantify the entanglement for state of the atom–field system, the von Neumann entropy is introduced, which is utilized as proper measure of the disorder and purity of a quantum state. In this work, we are going to examine the purity for the subsystem state acquired from the bipartite state expressed in Equation (17), $\hat{\rho}_{AF}(t) = |\psi(t)\rangle_{AF\ AF}\langle\psi(t)|$, by making a trace over one of the subsystem space. If $t = 0$, the bipartite system is in a pure state and therefore the subsystems possess the same entropy function throughout at subsequent times, i.e., $S = S_{\mathrm{A}}(t) = S_{\mathrm{F}}(t)$. To obtain the entanglement evolution for the bipartite system state, the quantum entropy of one subsystem was evaluated. Applying Equation (17) and $S_{\mathrm{F}}(t) = -\mathbf{Tr}(\hat{\rho}_{\mathrm{F}} \ln \hat{\rho}_{\mathrm{F}})$ the degree of entanglement is obtained as a function of the factors $C_{e,n}$ and $C_{g,n+1}$:

$$S_{\mathrm{F}}(t) = -\left[\Delta_{\mathrm{F}}^{+}\ln\left(\Delta_{\mathrm{F}}^{+}\right) + \Delta_{\mathrm{F}}^{-}\ln\left(\Delta_{\mathrm{F}}^{-}\right)\right], \tag{20}$$

where

$$\Delta_{\mathrm{F}}^{\pm} = \frac{1}{2}\left(1 \pm \sqrt{\left(\langle\beta|\beta\rangle - \langle\gamma|\gamma\rangle\right)^2 + 4|\langle\beta|\gamma|^2}\right), \tag{21}$$

with $\langle\beta|\beta\rangle = \sum_{n=0}^{\infty}|\mathbf{A}_{e,n}|^2$, $\langle\gamma|\gamma\rangle = \sum_{n=0}^{\infty}|\mathbf{A}_{g,n+1}|^2$ and $\langle\beta|\gamma\rangle = \langle\gamma|\beta\rangle^* = \sum_{n=0}^{\infty}\mathbf{A}_{e,n+1}^{*}$ $\mathbf{A}_{g,n+1}$. The von Neumann entropy changes from $S_V = 0$ for separable states to $S_V = 1$ for maximally entangled states.

The numerical results of the population inversion when the field is initially in a DCSCS have been displayed against the dimensionless time gt in Figures 3a and 4a for different values of the deformed parameter q with $|z| = 1$ and $\Phi = 0$. Figures 3a and 4b is for $j = 1$ and $j = 2$, respectively. The temporal evolution of the population inversion exhibits oscillations with probability amplitudes that depend on the values of q. Interestingly, the deformed parameter may be considered as an essential factor to control the inhibition decay of the excited state. Figure 5a indicates that the increase in the spin number j, the atomic population reveals the same structure but with fast oscillations during the time evolution.

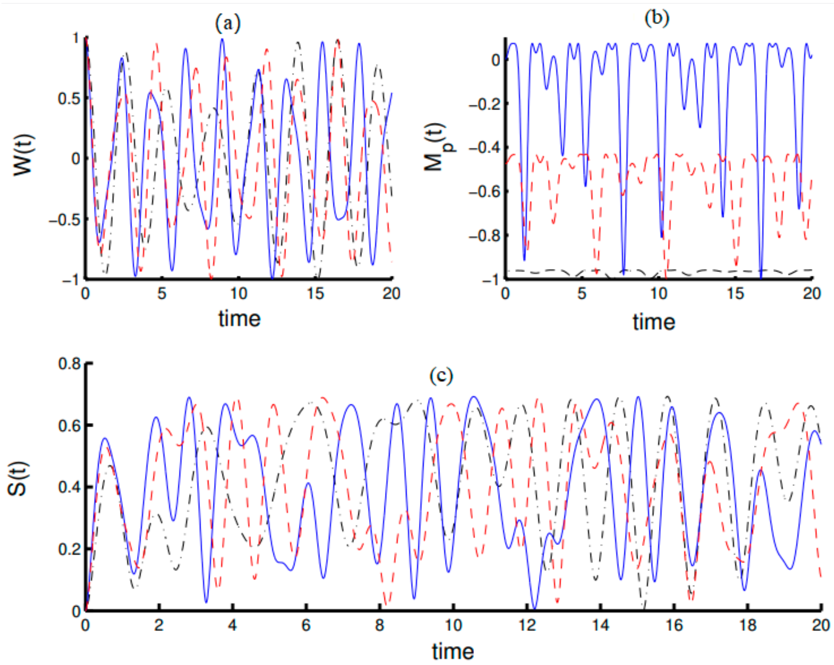

Figure 3. The dynamics of the different quantum quantifiers for specific values of the deformed parameter q with $|z| = 1$, $j = 1$, and $\Phi = 0$. (**a**) is for the atomic inversion, (**b**) is for the Mandel's parameter and (**c**) is for the von Neumann entropy. Blue (solid) line corresponds to $q = 1$, black (dotted-dashed) line corresponds to $q = 0.3$ and red (dashed) line corresponds to $q = 0.8$.

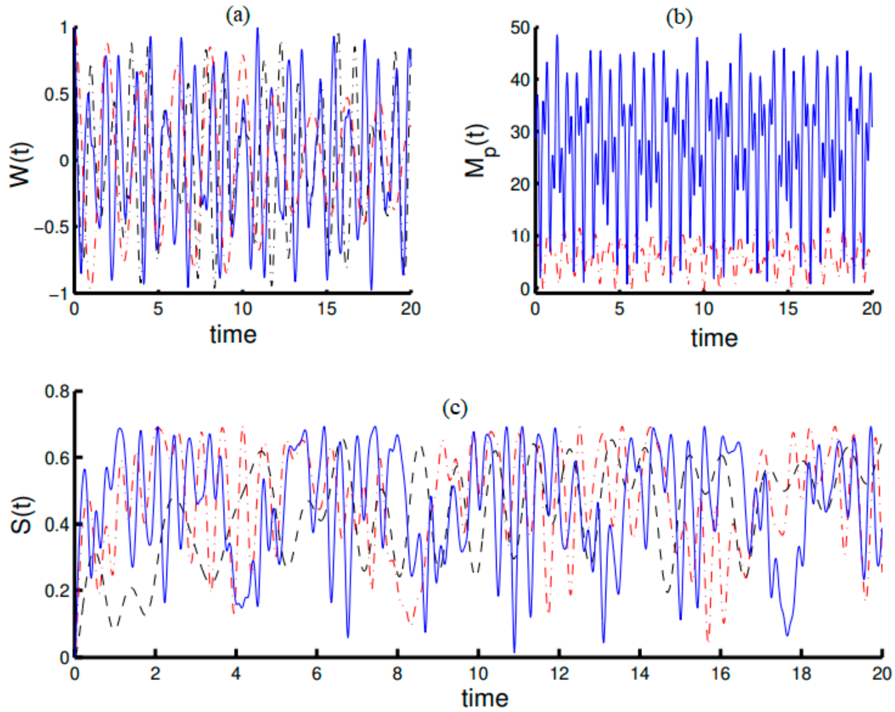

Figure 4. The dynamics of the different quantum quantifiers for specific values of the deformed parameter q with $|z| = 1$, $j = 2$, and $\Phi = 0$. (**a**) is for the atomic inversion, (**b**) and Figure 5d are for the Mandel's parameter and (**c**) is for the von Neumann entropy. Blue (solid) line corresponds to $q = 1$, black (dotted-dashed) line corresponds to $q = 0.3$ and red (dashed) line corresponds to $q = 0.8$.

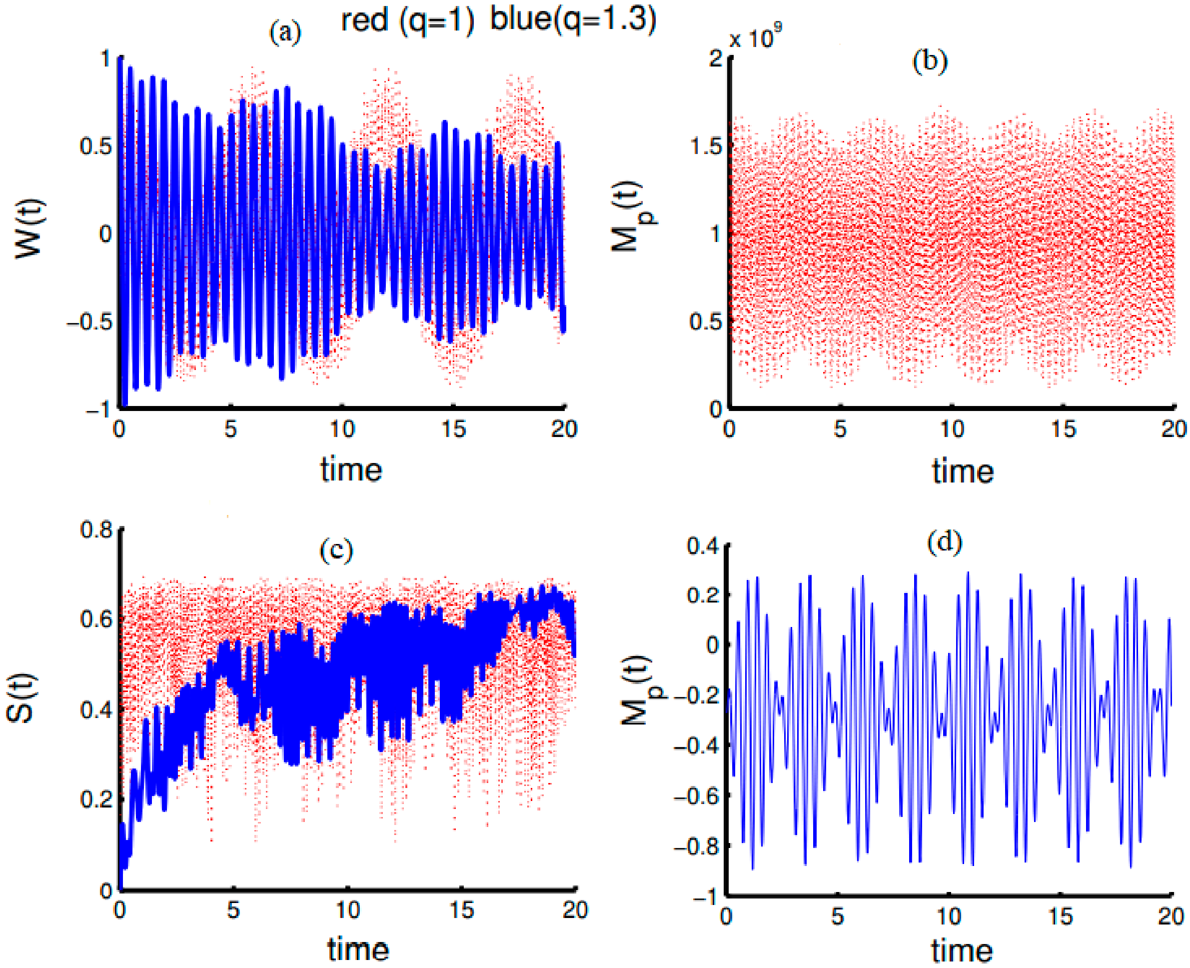

Figure 5. The dynamics of the different quantum quantifiers for specific values of the deformed parameter q with $|z| = 1$, $j = 6$, and $\Phi = 0$. (**a**) is for the atomic inversion, (**b**) and (**d**) are for the Mandel's parameter, and (**c**) is for the von Neumann entropy. Blue (solid) line corresponds to $q = 1$, black (dotted-dashed) line corresponds to $q = 0.3$ and red (dashed) line corresponds to $q = 0.8$.

The temporal evolution of the Mandel's parameter is displayed in Figures 3b and 4b against the dimensionless time gt for $j = 1$ and $j = 2$, respectively, considering different values of the deformed parameter q with $|z| = 1$ and $\Phi = 0$. In general, the parameter M_p exhibits oscillations that depend on the values of the spin number j and the values of the M_p are very sensitive to the deformed parameter q during the dynamics. For small values of spin, the Mandel's parameter takes negative values as the deformed parameter gets farther from one, exhibiting sub-Poissonian distribution. For the undeformed case, $q \to 1$, we can see that the photon statistics seem to be fluctuated around super-Poissonian and sub-Poissonian distributions during the time evolution. On the other hand, when the spin number increases, the Mandel's parameter tends to get positive values for different times, exhibiting super-Poissonian distribution of photons (see Figure 5b,d).

For an atomic system initially prepared in the upper level, the dynamical behavior of the nonlocal correlation in the atom–field system is analyzed when the quantized field is specified by a deformed spin cat state. This quantum correlation between the two-level atom and the field is generated through the interaction during the evolution. Figures 3c and 4c show the dynamics of the von-Neumann entropy against the dimensionless time μt for $j = 1$ and $j = 2$, respectively, with respect to the values of the deformed parameter q with $|z| = 1$ and $\Phi = 0$. It seems that the von Neumann entropy makes oscillatory behavior and its value depends on the choice of the parameter q. This means that this kind of fields may help to control the degree of the entanglement for the considered

JCM during the time evolution, where for some values of q, the entropy tends to attain its maximal value. On the other hand, the figures indicate that with increasing the spin number, the entropy exhibits very rapid oscillatory and for large values of the spin number j (see Figure 5c), the entropy behaves similarly to the case of superposition of the optical coherent states [42]. Finally, in Figure 6, we show the dynamical behavior of the different quantifiers for different values of the physical parameters with $\Phi = \pi/2$. It is clear that control of the quantifiers during the time-evolution can be made by a suitable choice of the phase parameter Φ defined in the deformed cat spin states.

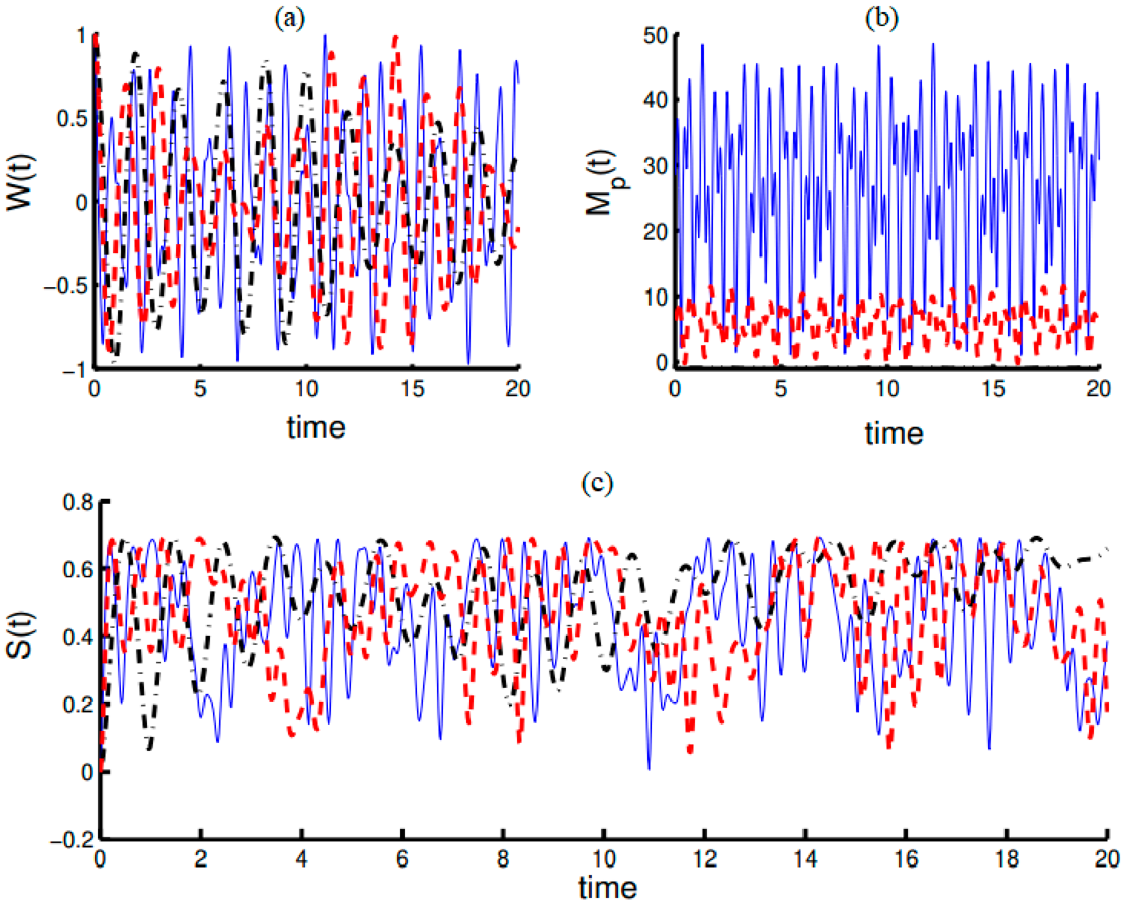

Figure 6. The dynamics of the different quantum quantifiers for specific values of the deformed parameter q with $|z| = 1$, $j = 1$, and $\Phi = \pi/2$. (**a**) is for the atomic inversion, (**b**) is for the Mandel's parameter and (**c**) is for the von Neumann entropy. Blue (solid) line corresponds to $q = 1$, black (dotted-dashed) corresponds to $q = 0.3$ and red (dashed) corresponds to $q = 0.8$.

5. Summary

In this manuscript, we have proposed a new kind of Schrödinger cat states introduced as a superposition of SCSs in the framework of noncommutative spaces. We have examined the nonclassical features of these noncommutative deformed states in terms of the main parameters of the physical parameters. We have suggested an appropriate quantum system for generating high amount of entanglement by a convenable control of the involved parameters. We have studied the dynamical behavior of the correlation and nonclassical properties for an atom–field system, where the field is initially described by a deformed cat spin state. In fact, we have shown in detail the dynamical behavior of the atomic population, Mandel's parameter, and entanglement in the bipartite system in terms of the deformation parameter, spin number, and phase parameter. It is known that the study of the physical properties of the interaction between the atomic and field is an important subject in quantum information and in optics. From that perspective, the obtained results show that this interaction in the presence of deformed cat spin states provide structures that are much richer than the non-deformed ones. A useful future study is the examination of the effect of decoherence in the presence of the deformed optical states for the case of open quantum system.

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