Research Article

Construction of 2-Peakon Solutions and Nonuniqueness for a Generalized mCH Equation

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For the generalized mCH equation, we construct a 2-peakon solution on both the line and the circle, and we can control the size of the initial data. The two peaks at different speeds move in the same direction and eventually collide. This phenomenon is that the solution at the collision time is consistent with another solitary peakon solution. By reversing the time, we get two new solutions with the same initial value and different values at the rest of the time, which means the nonuniqueness for the equation in Sobolev spaces H^s is proved for $s < 3/2$.

1. Introduction

The Camassa-Holm (CH) equation [1–3] is an integrable system with a bi-Hamiltonian structure, which is derived by Camassa and Holm using the asymptotic expansion in the Hamiltonian for Euler's equation. A special kind of weak solution for this equation describes the solitary wave at the peak, called peakons [4, 5], whose wave slope is discontinuous at the peak. The interactions between any number of peakons were described by the multipeakon solutions [6, 7], in the form of a linear superposition of peakons whose amplitude and velocity change with time.

In recent years, people's great interest in the research of Camassa-Holm (CH) equation has inspired people to explore the CH-type equation, especially the equations that admit peakons and mutipeakons. The CH, Degasperis-Procesi (DP) [8–13], modified CH (mCH) [14–19], and Novikov (NE) [20–22] equations are all integrable systems that admit peakons and mutipeakons. Of course, there are also some nonintegrable systems that admit peakons and mutipeakons, such as the b-family of equations [23], the modified b-family of equations [24], and the cubic ab-family of equations [25]. It is worth noting that the b-family of equations includes the CH equation and the DP equation, the modified b-family of equations includes the NE equation, and the cubic ab-family of equations includes the mCH equation and the NE equation.

With the development of research, great interest has been aroused in the uniqueness or posedness of solutions, setting initial value $u_0(x) \in H^s(\mathbb{R})$. The study of Li and Olver [26] shows that the CH equation is locally well posed in H^s for $s > 3/2$, and Byers [27] proved the ill-posedness for the CH equation in H^s when $s < 3/2$. Himonas, Grayshan, and Holliman [28] studied the ill-posedness for the DP equation. Himonas and Holliman [29] proved that the NE equation is well posed in H^s for $s > 3/2$. Himonas, Kenig, and Holliman [30] demonstrated the nonuniqueness for the NE equation in H^s when $s < 3/2$ by studying the collision of the peakons. Guo et al. [31] studied the ill-posedness for the CH, DP, and NE equations in critical spaces. Himonas and Mantzavinos [32] proved that the FORQ equation (also called mCH) is well posed in H^s for $s > 5/2$. The nonuniqueness results of Himonas and Holliman [33] show that solutions to the Cauchy problem for the FORQ equation are not unique in H^s when $s < 3/2$. At present, there is no theory to show the uniqueness for the FORQ equation in H^s when $3/2 \leq s \leq 5/2$. Holmes and Puri [34] discussed the nonuniqueness for the ab-family of equations. Himonas, Grayshan, and

Holliman [35] considered the ill-posedness for the b-family of equations in H^s for $s < 3/2$ when $b > 1$. On this basis, Novruzov [36] studied the ill-posedness for the b-family of equations when $b < 1$.

In this paper, we consider the Cauchy problem for a generalized mCH (gm- CH) equation which has the following form

$$m_t + \left(\left(u^2 - u_x^2 \right)^2 m \right)_x = 0, m = u - u_{xx}, \tag{1}$$

$$u(x,0) = u_0(x) \in H^s, t > 0, x \in X = \mathbb{R} \text{ or } \mathbb{T}. \tag{2}$$

This equation is obtained by Anco and Recio [37], by extending a Hamiltonian structure of the CH equation. Substituting $m = u - u_{xx}$ into the first equation of (1), it infers the following partial differential equation

$$u_t + u^4 u_x + \left(\frac{1}{5} u_x^5 - \frac{2}{3} u^2 u_x^3 \right) - \left(1 - \partial_x^2 \right)^{-1} \left(\frac{1}{5} u_x^5 - \frac{2}{3} u^2 u_x^3 \right)$$
$$+ \left(1 - \partial_x^2 \right)^{-1} \partial_x \left(\frac{4}{5} u^5 - \frac{1}{3} u u_x^4 + 2 u^3 u_x^2 \right) = 0. \tag{3}$$

The results of Anco and Recio [37] show that the gmCH equation admits peakon traveling wave solutions and multipeakon solutions. They studied the existence of the single peakon travelling solutions with $c \neq 0$ and classification of 2-peakon solutions. Recio and Anco [38] considered the conservation laws (energy, momentum, H^1-norm, etc.) of the gmCH equation, by modifying the general multiplier method combined with some tools from variational calculus. They also discussed the Hamiltonian structure and solitary traveling waves of the gmCH equation, by using the conservation laws. One remark is that the Hamiltonian structure for the family (1) corresponds to an energy conservation law that has a local density but a nonlocal flux.

Based on the conservation laws in [38], the Cauchy problem and nonuniqueness of the peakon solutions in this paper are studied. Under this premise, we obtain our main result, and its proof is closely related to the conservation of norms. And based on the existence of peakons in [37], we conduct the research on the peakon solutions. The difference is that we obtain the peakon traveling wave solutions by verifying the weak solution. The peakon traveling wave solutions on the line are given by

$$u(x,t) = a e^{-|x-ct|}, \quad \text{where} \quad c = \frac{11}{18} a^4. \tag{4}$$

On the circle, they are given by

$$u(x,t) = b \cdot \text{ch}(\zeta), \quad \zeta = \pi - [x - ct]_p, b^4$$
$$= \frac{c}{8/15 \text{sh}^4(\pi) + 4/3 \text{sh}^2(\pi) + 1}, \tag{5}$$

where $[.]_p$ is defined by

$$[x]_p = x - 2\pi \left[\frac{x}{2\pi} \right]. \tag{6}$$

On the other hand, the classification of 2-peakon solutions in [37] helps us construct 2-peakon solutions. In contrast to this, we construct a special 2-peakon solution based on the characteristics of the ODE system and study the collision of peakons. The result is summarized in the following theorem.

Theorem 1. *Solutions to the Cauchy problem for the gmCH equation (1) are not unique in Sobolev spaces H^s when $s < 3/2$.*

The rest is organized as follows. In Section 2, we study the ODE systems that the 2-peakon solutions of the gmCH Equation (1) need to satisfy. In Section 3, we give the proof of Theorem 1 on the line by constructing a 2-peakon solution. In Section 4, we prove Theorem 1 on the circle.

2. 2-Peakon on the Line and the Circle

In [37], Recio and Anco studied the multipeakon solutions on the line, and they proved the following result.

Theorem 2 (see [37]). *The nonperiodic 2-peakon*

$$u(x,t) = p_1(t) e^{-|x-q_1(t)|} + p_2(t) e^{-|x-q_2(t)|}, (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{7}$$

is a solution to Equation (1) if its positions q_1, q_2 and momenta p_1, p_2 satisfy

$$p_1' = 0, q_1' = \frac{8}{15} p_1^2 \left(p_1^2 + 5 p_1 p_2 e^{-|q_1-q_2|} + 10 p_2^2 e^{-2|q_1-q_2|} \right),$$
$$p_2' = 0, q_2' = \frac{8}{15} p_2^2 \left(p_2^2 + 5 p_1 p_2 e^{-|q_1-q_2|} + 10 p_1^2 e^{-2|q_1-q_2|} \right). \tag{8}$$

Now, we consider the 2-peakon system on the circle, based on the methods in [25].

Theorem 3. *The periodic 2-peakon*

$$u(x,t) = p_1(t) \text{ch} \left([x - q_1(t)]_p - \pi \right) + p_2(t) \text{ch} \left([x - q_2(t)]_p - \pi \right) \tag{9}$$

is a solution to Equation (1) if its positions q_1, q_2. and momenta p_1, p_2 satisfy

$$q_1' = \frac{4}{5}p_1^2\left[1 + \frac{2}{3}sh^2(\pi)\right]\left[p_1^2 ch^2(\pi) + 5p_1p_2 ch(\pi)ch\left([q_1 - q_2]_p - \pi\right)\right]$$
$$+ \frac{1}{5}p_1^4 + p_2^4 + 4p_1p_2^3 ch(\pi)ch\left([q_1 - q_2]_p - \pi\right)$$
$$+ 2p_1^2 p_2^2\left[\frac{8}{3}ch^2(\pi)ch^2\left([q_1 - q_2]_p - \pi\right) - \frac{2}{3}ch^2\left([q_1 - q_2]_p - \pi\right) + 1\right],$$
$$(10)$$

$$q_2' = \frac{4}{5}p_2^2\left[1 + \frac{2}{3}sh^2(\pi)\right]\left[p_2^2 ch^2(\pi) + 5p_1p_2 ch(\pi)ch\left([q_2 - q_1]_p - \pi\right)\right]$$
$$+ \frac{1}{5}p_2^4 + p_1^4 + 4p_1^3 p_2 ch(\pi)ch\left([q_2 - q_1]_p - \pi\right)$$
$$+ 2p_1^2 p_2^2\left[\frac{8}{3}ch^2(\pi)ch^2\left([q_2 - q_1]_p - \pi\right) - \frac{2}{3}ch^2\left([q_2 - q_1]_p - \pi\right) + 1\right],$$
$$(11)$$

$$p_1' = 0, p_2' = 0, \tag{12}$$

where $[\cdot]_p$ is defined as in (6).

Proof. We can rewrite the equation (1) as the following equivalent form

$$\left(1 - \partial_x^2\right)u_t + \left(1 - \frac{1}{5}\partial_x^2\right)\left(\partial_x(u^5) + u_x^5\right) + \partial_x^2\left(\frac{2}{3}u^2 u_x^3\right)$$
$$+ \partial_x\left(2u^3 u_x^2 - \frac{1}{3}uu_x^4\right) - u_x^5 = 0.$$
$$(13)$$

Let $\varphi \in C^\infty(\mathbb{T})$ be any smooth periodic test function on $\mathbb{T}$, and

$$u_j(x, t) = p_j(t)ch\left([x - q_j(t)]_p - \pi\right), j = 1, 2, \tag{14}$$

which causes the periodic 2-peakon solution (9) to be rewritten as $u(x, t) = u_1(x, t) + u_2(x, t)$. So, we have

$$\left\langle\left(1 - \partial_x^2\right)u_t, \varphi\right\rangle = \left\langle\left(1 - \partial_x^2\right)u_{1t}, \varphi\right\rangle + \left\langle\left(1 - \partial_x^2\right)u_{2t}, \varphi\right\rangle. \tag{15}$$

Firstly, we calculate $\langle\partial_x u_j, \varphi\rangle$. Note that when $q_j \in (2k\pi, 2(k+1)\pi)$, $k \in \mathbb{N}$, we have $[q_j/2\pi] = k$ and $[-q_j/2\pi] = -(k+1)$, which leads to $[2\pi - q_j/2\pi] = -k$. Let $q_j^* = q_j - 2\pi[q_j/2\pi]$. Obviously, $q_j^* \in (0, 2\pi)$. Since $[x - q_j/2\pi] = -(k+1)$ for $0 < x < q_j^*$ and $[x - q_j/2\pi] = -k$ for $q_j^* < x < 2\pi$, it obtains by integrating by parts

$$\langle\partial_x u_j, \varphi\rangle = -\int_0^{q_j^*} p_j ch\left(x - q_j + (2k+1)\pi\right)d\varphi$$
$$- \int_{q_j^*}^{2\pi} p_j ch\left(x - q_j + (2k-1)\pi\right)d\varphi$$
$$= -p_j ch(\pi)\varphi\left(q_j^*\right) + p_j ch\left(-q_j + (2k+1)\pi\right)\varphi(0)$$
$$+ p_j ch(-\pi)\varphi\left(q_j^*\right) - p_j ch\left(-q_j + (2k+1)\pi\right)\varphi(2\pi)$$
$$+ \left\langle p_j sh\left([x - q_j]_p - \pi\right), \varphi\right\rangle.$$
$$(16)$$

Since $ch(x) = ch(-x)$ and $\varphi(x) = \varphi(x + 2\pi)$,

$$\langle\partial_x u_j, \varphi\rangle = \left\langle p_j sh\left([x - q_j]_p - \pi\right), \varphi\right\rangle. \tag{17}$$

On the other hand, when $q_j = 2k\pi$, $k \in \mathbb{N}$, we find $[x - q_j/2\pi] = -k$ for $x \in (0, 2\pi)$. It follows that

$$\langle\partial_x u_j, \varphi\rangle = -\int_0^{2\pi} p_j ch\left([x - q_j]_p - \pi\right)\varphi' dx$$
$$= \left\langle p_j sh\left([x - q_j]_p - \pi\right), \varphi\right\rangle,$$
$$(18)$$

along with (17) leads to $\langle\partial_x u_j, \varphi\rangle = \langle p_j sh([x - q_j]_p - \pi), \varphi\rangle$, for all $q_j \geq 0$. Analogously, since $sh(\cdot)$ is odd, $\langle\partial_x^2 u_j, \varphi\rangle = -2p_j sh(\pi)\varphi(q_j^*) + \langle u_j, \varphi\rangle$, which means that $\langle(1 - \partial_x^2)u_j, \varphi\rangle = 2p_j sh(\pi)\varphi(q_j^*)$. Moreover, we find

$$\left\langle\left(1 - \partial_x^2\right)u_t, \varphi\right\rangle = 2sh(\pi)\left[p_1'\varphi\left(q_1^*\right) + p_1 q_1'\varphi'\left(q_1^*\right) + p_2'\varphi\left(q_2^*\right) + p_2 q_2'\varphi'\left(q_2^*\right)\right]. \tag{19}$$

Now, we conclude $\langle-\partial_x^2(u^4 u_x), \varphi\rangle$. We use $u(x, t) = u_1(x, t) + u_2(x, t)$ to get

$$\left\langle-\partial_x^2\left(u^4 u_x\right), \varphi\right\rangle = -\left\langle u^4 u_x, \varphi''\right\rangle$$
$$= -\langle u_1^4 u_{1x} + 4u_1^3 u_{1x}u_2 + 6u_1^2 u_{1x}u_2^2 + 4u_1 u_{1x}u_2^3 + u_{1x}u_2^4$$
$$+ u_2^4 u_{2x} + 4u_1 u_2^3 u_{2x} + 6u_1^2 u_2^2 u_{2x} + 4u_1^3 u_2 u_{2x} + u_1^4 u_{2x}, \varphi''\rangle.$$
$$(20)$$

Since $[-q_j/2\pi] = -(k+1)$ for $q_j \in (2k\pi, 2(k+1)\pi)$, where $k \in \mathbb{N}$, we find $[0 - q_j]_p - \pi = -q_j + 2(k+1)\pi$ and $[2\pi - q_j]_p - \pi = -q_j + 2(k+1)\pi$, which combined with integration by parts give that

$$\left\langle u_1^4 u_{1x}, \varphi'' \right\rangle = \left\langle p_1^5 \text{ch}^4 \left([x-q_1]_p - \pi\right) \text{sh}\left([x-q_1]_p - \pi\right), \varphi'' \right\rangle$$
$$= p_1^5 \left[\int_0^{q_1^*} \text{ch}^4(x - q_1 + (2k+1)\pi)\text{sh}(x - q_1 + (2k+1)\pi)\varphi'' dx \right.$$
$$\left. + \int_{q_1^*}^{2\pi} \text{ch}^4(x - q_1 + (2k-1)\pi)\text{sh}(x - q_1 + (2k-1)\pi)\varphi'' dx \right]$$
$$= 2p_1^5 \text{ch}^4(\pi)\text{sh}(\pi)\varphi'(q_1^*)$$
$$+ \left\langle 12u_1^2 u_{1x}^3 + 12u_1^3 u_{1x} u_{1xx} + u_1^4 u_{1xxx}, \varphi \right\rangle,$$

$$\left\langle u_2^4 u_{2x}, \varphi'' \right\rangle = \left\langle p_2^5 \text{ch}^4 \left([x-q_2]_p - \pi\right) \text{sh}\left([x-q_2]_p - \pi\right), \varphi'' \right\rangle$$
$$= 2p_2^5 \text{ch}^4(\pi)\text{sh}(\pi)\varphi'(q_2^*)$$
$$+ \left\langle 12u_2^2 u_{2x}^3 + 12u_2^3 u_{2x} u_{2xx} + u_2^4 u_{2xxx}, \varphi \right\rangle. \tag{21}$$

Without loss of generality, we assume $q_1 \le q_2$. So, we have

$$\left\langle 4u_1^3 u_{1x} u_2, \varphi'' \right\rangle = 4p_1^4 p_2 \left(\int_0^{q_1^*} + \int_{q_1^*}^{q_2^*} + \int_{q_2^*}^{2\pi} \right) \text{ch}^3 \left([x-q_1]_p - \pi\right)$$
$$\cdot \text{sh}\left([x-q_1]_p - \pi\right)\text{ch}\left([x-q_2]_p - \pi\right)\varphi'' dx$$
$$= 8p_1^4 p_2 \text{sh}(\pi) \left[\text{ch}^3(\pi)\text{ch}\left([q_1^* - q_2]_p - \pi\right)\varphi'(q_1^*) \right.$$
$$- \text{ch}^3(\pi)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*)$$
$$\left. - \text{ch}^3 \left([q_2^* - q_1]_p - \pi\right)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*) \right]$$
$$+ \left\langle 24u_1 u_{1x}^3 u_2 + 36u_1^2 u_{1x} u_{1xx} u_2 + 24u_1^2 u_{1x}^2 u_{2x} \right.$$
$$\left. + 8u_1^3 u_{1xx} u_{2x} + 4u_1^3 u_{1x} u_{2xx} + 4u_1^3 u_{1xxx} u_2, \varphi \right\rangle, \tag{22}$$

$$\left\langle 6u_1^2 u_{1x} u_2^2, \varphi'' \right\rangle = 6p_1^3 p_2^2 \left(\int_0^{q_1^*} + \int_{q_1^*}^{q_2^*} + \int_{q_2^*}^{2\pi} \right) \text{ch}^2 \left([x-q_1]_p - \pi\right)$$
$$\cdot \text{sh}\left([x-q_1]_p - \pi\right)\text{ch}^2 \left([x-q_2]_p - \pi\right)\varphi'' dx$$
$$= 12p_1^3 p_2^2 \text{sh}(\pi) \left[\text{ch}^2(\pi)\text{ch}^2 \left([q_1^* - q_2]_p - \pi\right)\varphi'(q_1^*) \right.$$
$$- 2\text{ch}^2(\pi)\text{ch}\left([q_1^* - q_2]_p - \pi\right)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*)$$
$$\left. - 2\text{ch}(\pi)\text{ch}^2 \left([q_2^* - q_1]_p - \pi\right)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*) \right]$$
$$+ \left\langle 12u_{1x}^3 u_2^2 + 36u_1 u_{1x} u_{1xx} u_2^2 + 48u_1 u_{1x}^2 u_2 u_{2x} \right.$$
$$+ 24u_1^2 u_{1xx} u_2 u_{2x} + 12u_1^2 u_{1x} u_{2x}^2 + 12u_1^2 u_{1x} u_2 u_{2xx}$$
$$\left. + 6u_1^2 u_{1xxx} u_2^2, \varphi \right\rangle. \tag{23}$$

Similar to (22), we obtain

$$\left\langle 4u_1 u_{1x} u_2^3, \varphi'' \right\rangle = 4p_1^2 p_2^3 \left(\int_0^{q_1^*} + \int_{q_1^*}^{q_2^*} + \int_{q_2^*}^{2\pi} \right) \text{ch}\left([x-q_1]_p - \pi\right)$$
$$\cdot \text{sh}\left([x-q_1]_p - \pi\right)\text{ch}^3 \left([x-q_2]_p - \pi\right)\varphi'' dx$$
$$= 8p_1^2 p_2^3 \text{sh}(\pi) \left[\text{ch}(\pi)\text{ch}^3 \left([q_1^* - q_2]_p - \pi\right)\varphi'(q_1^*) \right.$$
$$- 3\text{ch}(\pi)\text{ch}^2 \left([q_1^* - q_2]_p - \pi\right)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*)$$
$$\left. - 3\text{ch}^2(\pi)\text{ch}\left([q_2^* - q_1]_p - \pi\right)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*) \right]$$
$$+ \left\langle 12u_{1x} u_{1xx} u_2^3 + 24u_{1x}^2 u_2^2 u_{2x} + 24u_1 u_{1xx} u_2^2 u_{2x} \right.$$
$$\left. + 24u_1 u_{1x} u_2 u_{2x}^2 + 12u_1 u_{1x} u_2^2 u_{2xx} + 4u_1 u_{1xxx} u_2^3, \varphi \right\rangle,$$

$$\left\langle u_{1x} u_2^4, \varphi'' \right\rangle = p_1 p_2^4 \left(\int_0^{q_1^*} + \int_{q_1^*}^{q_2^*} + \int_{q_2^*}^{2\pi} \right) \text{sh}\left([x-q_1]_p - \pi\right)\text{ch}^4 \left([x-q_2]_p - \pi\right)\varphi'' dx$$
$$= 2p_1 p_2^4 \text{sh}(\pi) \left[\text{ch}^4 \left([q_1^* - q_2]_p - \pi\right)\varphi'(q_1^*) \right.$$
$$- 4\text{ch}^3 \left([q_1^* - q_2]_p - \pi\right)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*)$$
$$\left. - 4\text{ch}^3(\pi)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*) \right]$$
$$+ \left\langle u_{1xxx} u_2^4 + 8u_{1xx} u_2^3 u_{2x} + 12u_{1x} u_2^2 u_{2x}^2 + 4u_{1x} u_2^3 u_{2xx}, \varphi \right\rangle. \tag{24}$$

Although the above calculation is made by assuming $q_1 \le q_2$, the result is also valid to any q_1 and q_2. We next compute

$$\left\langle 4u_1 u_2^3 u_{2x}, \varphi'' \right\rangle = 8p_1 p_2^4 \text{sh}(\pi) \left[\text{ch}^3(\pi)\text{ch}\left([q_2^* - q_1]_p - \pi\right)\varphi'(q_2^*) \right.$$
$$- \text{ch}^3(\pi)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*)$$
$$\left. - \text{ch}^3 \left([q_1^* - q_2]_p - \pi\right)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*) \right]$$
$$+ \left\langle 24u_1 u_2 u_{2x}^3 + 36u_1 u_2^2 u_{2x} u_{2xx} + 24u_{1x} u_2^2 u_{2x}^2 \right.$$
$$\left. + 8u_{1x} u_2^3 u_{2xx} + 4u_{1xx} u_2^3 u_{2x} + 4u_1 u_2^3 u_{2xxx}, \varphi \right\rangle, \tag{25}$$

$$\left\langle 6u_1^2 u_2^2 u_{2x}, \varphi'' \right\rangle = 12p_1^2 p_2^3 \text{sh}(\pi) \left[\text{ch}^2(\pi)\text{ch}^2 \left([q_2^* - q_1]_p - \pi\right)\varphi'(q_2^*) \right.$$
$$- 2\text{ch}^2(\pi)\text{ch}\left([q_2^* - q_1]_p - \pi\right)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*)$$
$$\left. - 2\text{ch}(\pi)\text{ch}^2 \left([q_1^* - q_2]_p - \pi\right)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*) \right]$$
$$+ \left\langle 12u_1^2 u_{2x}^3 + 36u_1^2 u_2 u_{2x} u_{2xx} + 48u_1 u_{1x} u_2 u_{2x}^2 \right.$$
$$+ 24u_1 u_{1x} u_2^2 u_{2xx} + 12u_{1x}^2 u_2^2 u_{2x} + 12u_1 u_{1xx} u_2^2 u_{2x}$$
$$\left. + 6u_1^2 u_2^2 u_{2xxx}, \varphi \right\rangle. \tag{26}$$

$$\left\langle 4u_1^3 u_2 u_{2x}, \varphi'' \right\rangle = 8p_1^3 p_2^2 \text{sh}(\pi) \left[\text{ch}(\pi)\text{ch}^3 \left([q_2^* - q_1]_p - \pi\right)\varphi'(q_2^*) \right.$$
$$- 3\text{ch}(\pi)\text{ch}^2 \left([q_2^* - q_1]_p - \pi\right)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*)$$
$$\left. - 3\text{ch}^2(\pi)\text{ch}\left([q_1^* - q_2]_p - \pi\right)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*) \right]$$
$$+ \left\langle 12u_1^2 u_{2x} u_{2xx} + 24u_1^2 u_{1x} u_{2x}^2 + 24u_1^2 u_{1x} u_2 u_{2xx} \right.$$
$$\left. + 24u_1 u_{1x}^2 u_2 u_{2x} + 12u_1^2 u_{1xx} u_2 u_{2x} + 4u_1^3 u_2 u_{2xxx}, \varphi \right\rangle, \tag{27}$$

$$\left\langle u_1^4 u_{2x}, \varphi'' \right\rangle = 2p_1^4 p_2 \text{sh}(\pi) \left[\text{ch}^4 \left([q_2^* - q_1]_p - \pi\right)\varphi'(q_2^*) \right.$$
$$- 4\text{ch}^3 \left([q_2^* - q_1]_p - \pi\right)\text{sh}\left([q_2^* - q_1]_p - \pi\right)\varphi(q_2^*)$$
$$\left. - 4\text{ch}^3(\pi)\text{sh}\left([q_1^* - q_2]_p - \pi\right)\varphi(q_1^*) \right]$$
$$+ \left\langle u_1^4 u_{2xxx} + 8u_1^3 u_{1x} u_{2xx} + 12u_1^2 u_{1x}^2 u_{2x} + 4u_1^3 u_{1xx} u_{2x}, \varphi \right\rangle. \tag{28}$$

It follows from (20)-(28) that

$$\left\langle -\partial_x^2 (u^4 u_x), \varphi \right\rangle = -2p_1 \mathrm{sh}(\pi)\varphi'(q_1^*)\left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]^4$$
$$- 2p_2\mathrm{sh}(\pi)\varphi'(q_2^*)\left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]^4$$
$$+ 16p_1 p_2\mathrm{sh}(\pi)\varphi(q_1^*)\mathrm{sh}\left([q_1^* - q_2]_p - \pi\right)$$
$$\cdot \left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]^3 + 16p_1 p_2\mathrm{sh}(\pi)\varphi(q_2^*)\mathrm{sh}$$
$$\left([q_2^* - q_1]_p - \pi\right)\left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]^3$$
$$- \left\langle u^4 u_{xxx} + 12u^3 u_x u_{xx} + 12u^2 u_x^3, \varphi \right\rangle. \tag{29}$$

Similar to (29), it infers

$$\left\langle -\partial_x^2 \left(\frac{1}{5}u^5\right), \varphi \right\rangle = -2p_1\mathrm{sh}(\pi)\varphi'(q_1^*)$$
$$\cdot \left(-\frac{4}{5}p_1^4\mathrm{sh}^4(\pi) + \left[p_1^2\mathrm{sh}^2(\pi) + p_2^2\mathrm{sh}^2\left([q_1^* - q_2]_p - \pi\right)\right]^2\right)$$
$$- 2p_2\mathrm{sh}(\pi)\varphi'(q_2^*)\left(-\frac{4}{5}p_2^4\mathrm{sh}^4(\pi)\right.$$
$$\left. + \left[p_2^2\mathrm{sh}^2(\pi) + p_1^2\mathrm{sh}^2\left([q_2^* - q_1]_p - \pi\right)\right]^2\right)$$
$$+ 8p_1 p_2\mathrm{sh}(\pi)\varphi(q_1^*)\mathrm{sh}\left([q_1^* - q_2]_p - \pi\right)$$
$$\cdot \left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]$$
$$\cdot \left[p_1^2\mathrm{sh}^2(\pi) + p_2^2\mathrm{sh}^2\left([q_1^* - q_2]_p - \pi\right)\right]$$
$$+ 8p_1 p_2\mathrm{sh}(\pi)\varphi(q_2^*)\mathrm{sh}\left([q_2^* - q_1]_p - \pi\right)$$
$$\cdot \left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]$$
$$\cdot \left[p_2^2\mathrm{sh}^2(\pi) + p_1^2\mathrm{sh}^2\left([q_2^* - q_1]_p - \pi\right)\right]$$
$$- \left\langle u_x^4 u_{xxx} + 4u_x^3 u_{xx}^2, \varphi \right\rangle,$$

$$\left\langle \partial_x^2 \left(\frac{2}{3}u^2 u_x^3\right), \varphi \right\rangle = 2p_1\mathrm{sh}(\pi)\varphi'(q_1^*)$$
$$\cdot \left(\frac{2}{3}\left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]^2\right.$$
$$\left. \cdot \left[p_1^2\mathrm{sh}^2(\pi) + 3p_2^2\mathrm{sh}^2\left([q_1^* - q_2]_p - \pi\right)\right]\right)$$
$$+ 2p_2\mathrm{sh}(\pi)\varphi'(q_2^*) \cdot \left(\frac{2}{3}\left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]^2\right.$$
$$\left. \cdot \left[p_2^2\mathrm{sh}^2(\pi) + 3p_1^2\mathrm{sh}^2\left([q_2^* - q_1]_p - \pi\right)\right]\right)$$
$$- 8p_1 p_2\mathrm{sh}(\pi)\varphi(q_1^*)\mathrm{sh}\left([q_1^* - q_2]_p - \pi\right)$$
$$\cdot \left(\left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]^3\right.$$
$$+ \frac{4}{3}\left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]$$
$$\left. \cdot \left[p_1^2\mathrm{sh}^2(\pi) + p_2^2\mathrm{sh}^2\left([q_1^* - q_2]_p - \pi\right)\right]\right)$$
$$- 8p_1 p_2\mathrm{sh}(\pi)\varphi(q_2^*)\mathrm{sh}\left([q_2^* - q_1]_p - \pi\right)$$
$$\cdot \left(\left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]^3\right.$$
$$+ \frac{4}{3}\left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]$$
$$\left. \cdot \left[p_2^2\mathrm{sh}^2(\pi) + p_1^2\mathrm{sh}^2\left([q_2^* - q_1]_p - \pi\right)\right]\right)$$
$$+ \left\langle \frac{4}{3}u_x^5 + \frac{28}{3}uu_x^3 u_{xx} + 2u^2 u_x^2 u_{xxx} + 4u^2 u_x u_{xx}^2, \varphi \right\rangle \tag{30}$$

with

$$\left\langle \partial_x \left(2u^3 u_x^2\right), \varphi \right\rangle - 8p_1 p_2\mathrm{sh}(\pi)\varphi(q_1^*)\mathrm{sh}\left([q_1^* - q_2]_p - \pi\right)$$
$$\cdot \left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]^3$$
$$- 8p_1 p_2\mathrm{sh}(\pi)\varphi(q_2^*)\mathrm{sh}\left([q_2^* - q_1]_p - \pi\right)$$
$$\left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]^3 + \left\langle 4u^3 u_x u_{xx} + 6u^2 u_x^3, \varphi \right\rangle, \tag{31}$$

$$\left\langle \partial_x \left(\frac{1}{3}uu_x^4\right), \varphi \right\rangle = \frac{8}{3}p_1 p_2\mathrm{sh}(\pi)\varphi(q_1^*)\mathrm{sh}\left([q_1^* - q_2]_p - \pi\right)$$
$$\cdot \left[p_1\mathrm{ch}(\pi) + p_2\mathrm{ch}\left([q_1^* - q_2]_p - \pi\right)\right]$$
$$\cdot \left[p_1^2\mathrm{sh}^2(\pi) + p_2^2\mathrm{sh}^2\left([q_1^* - q_2]_p - \pi\right)\right]$$
$$+ \frac{8}{3}p_1 p_2\mathrm{sh}(\pi)\varphi(q_2^*)\mathrm{sh}\left([q_2^* - q_1]_p - \pi\right)$$
$$\cdot \left[p_2\mathrm{ch}(\pi) + p_1\mathrm{ch}\left([q_2^* - q_1]_p - \pi\right)\right]$$
$$\cdot \left[p_2^2\mathrm{sh}^2(\pi) + p_1^2\mathrm{sh}^2\left([q_2^* - q_1]_p - \pi\right)\right]$$
$$- \left\langle \frac{1}{3}u_x^5 + \frac{4}{3}uu_x^3 u_{xx}, \varphi \right\rangle. \tag{32}$$

Substituting (29)-(32) into the equation (13), also noting $[q_1^* - q_2]_p = [q_1 - q_2]_p$ and $[q_2^* - q_1]_p = [q_2 - q_1]_p$, which is caused by the fact that $[q_j^*/2\pi] = 0$ for $q_j^* \in [0, 2\pi)$, we obtain the system (10). □

3. Nonuniqueness on the Line

In this section, we use the ODE system (8) to prove Theorem 1 on the line. To do this, we take a 2-peakon solution of the form (7). From the first two items of the system (8), $p_1(t) = p_1(0)$ and $p_2(t) = p_2(0)$ are obvious. At the same time, we have $q_1' = q_2'$ if we take the symmetric initial data, $|p_1(0)| = |p_2(0)|$. The two peaks move at the same speed which means there is no collision. Therefore, we choose the following initial data

$$p_1(0) = -(b + \delta), \quad q_1(0) = 0,$$
$$p_2(0) = b, \quad\quad\quad q_2(0) = a, \tag{33}$$

with $a > 0$ and $b^2 + (b + \delta)^2 > 5b(b + \delta)$, where $b + \delta > b > 0$. The selection of these initial data is summarized in Figure 1.

According to (33), $p_1(t) = -(b + \delta)$ and $p_2(t) = b$ are obtained. We introduce the symbol q to represent the difference between the positions of the two peakons, in
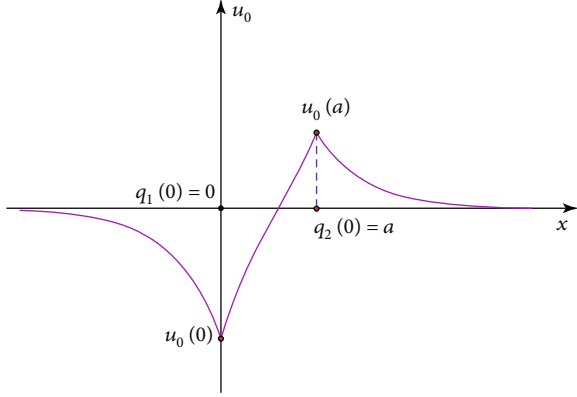
FIGURE 1: Initial profile $u_0(x)$.

other words, $q \doteq q_2 - q_1$. It follows from the ODE system (8) that

$$q' = \frac{8}{15}\left(p_2^4 - p_1^4\right)\left(1 + \alpha e^{-q}\right) = \frac{8}{15}\left(b^4 - (b + \delta)^4\right)\left(1 + \alpha e^{-q}\right) < 0, \tag{34}$$

where $-1 < \alpha = 5 p_1 p_2 / p_1^2 + p_2^2 = -5b(b + \delta)/b^2 + (b + \delta)^2 < 0$. Integrating (34), we calculate

$$\ln \frac{e^q + \alpha}{e^a + \alpha} = \frac{8}{15}\left(b^4 - (b + \delta)^4\right)t. \tag{35}$$

Since $q' < 0$ and $q_0 = q_2(0) - q_1(0) = a > 0$, we obtain a collision and a positive collision time when $q(t) = 0$. Using the symbol τ for the collision time, from (35), we find

$$\tau = \frac{15}{8\left(b^4 - (b + \delta)^4\right)} \ln \frac{1 + \alpha}{e^a + \alpha}. \tag{36}$$

Applying expressions $q_\tau \doteq \lim_{t \to \tau^-} q_1(t) = \lim_{t \to \tau^-} q_2(t)$ and v $(x) \doteq -\delta e^{-|x - q_\tau|}$ to define the collision location q_τ and the collision function v, we get the following proposition.

Proposition 4. The H^s limit of u as t approaches τ is v, or

$$\lim_{t \to \tau^-} \|u(t) - v\|_{H^s} = 0. \tag{37}$$

Proof. We compute the Fourier transform of u and v, which is denoted by

$$\widehat{u}(\xi, t) = \frac{2 p_1 e^{-i\xi q_1}}{1 + \xi^2} + \frac{2 p_2 e^{-i\xi q_2}}{1 + \xi^2} = \frac{-2(b + \delta)e^{-i\xi q_1}}{1 + \xi^2} + \frac{2 b e^{-i\xi q_2}}{1 + \xi^2}, \tag{38}$$

$$\widehat{v}(\xi) = \frac{-2\delta e^{-i\xi q_\tau}}{1 + \xi^2}. \tag{39}$$

Combining (38) and (39), we have

$$\lim_{t \to \tau^-} \|u(t) - v\|_{H^s}^2$$
$$= \lim_{t \to \tau^-} 4 \int_{\mathbb{R}} \left(1 + \xi^2\right)^{s-2} \left| b e^{-i\xi q_2} - (b + \delta)e^{-i\xi q_1} + \delta e^{-i\xi q_\tau} \right|^2 d\xi. \tag{40}$$

Notice that the equation inside the absolute value can be scaled up to

$$\left| b e^{-i\xi q_2} - (b + \delta)e^{-i\xi q_1} + \delta e^{-i\xi q_\tau} \right|$$
$$\leq (b + \delta)\left| e^{-i\xi q_1} \right| + b \left| e^{-i\xi q_2} \right| + \delta \left| e^{-i\xi q_\tau} \right| = 2(b + \delta). \tag{41}$$

Let $f(\xi) = 4(b + \delta)^2 (1 + \xi^2)^{s-2}$. There is no doubt that f is integrable when $s < 3/2$, and f dominates the original integrand, which means, we can apply the dominated convergence theorem and put the limit inside the integral. So, we get

$$\lim_{t \to \tau^-} 4 \int_{\mathbb{R}} \left(1 + \xi^2\right)^{s-2} \left| b e^{-i\xi q_2} - (b + \delta)e^{-i\xi q_1} + \delta e^{-i\xi q_\tau} \right|^2 d\xi$$
$$= 4 \int_{\mathbb{R}} \left(1 + \xi^2\right)^{s-2} \left| b e^{-i\xi q_2(\tau)} - (b + \delta)e^{-i\xi q_1(\tau)} + \delta e^{-i\xi q_\tau} \right|^2 d\xi = 0. \tag{42}$$

Proposition 4 is proven.   □

Proof of Theorem 1. (On the line). In view of the 2-peakon solution we constructed, we need to construct a traveling wave solution $w(x, t)$ that satisfies

$$w(x, \tau) = u(x, \tau) = v(x). \tag{43}$$

Reviewing the system (8), we take the following data

$$p_1(0) = -\delta, \quad q_1(0) = \beta,$$
$$p_2 = 0, \quad q_2(0) = 0. \tag{44}$$

The system is simplified and easy to be solved as

$$p_1(t) = -\delta, q_1(t) = \frac{8}{15}\delta^4 t + \beta. \tag{45}$$

We introduce the symbol q_1^u to represent q_1 corresponding to u and q_1^w to represent q_1 corresponding to w. It follows from (43) that $q_1^u(\tau) = q_1^w(\tau)$. So, we find $\beta = q_1^u(\tau)$ $- (\delta^4/(b^4 - (b + \delta)^4)) \ln (1 + \alpha/e^a + \alpha)$,which makes the construction complete. In order to prove the nonuniqueness of the solution, we define two new solutions $\tilde{u}(x, t) \doteq u(-x, -t + \tau), \tilde{w}(x, t) \doteq w(-x, -t + \tau)$. Since $u(x, t)$ and $w(x, t)$ are two solutions to (1), $u(-x, -t)$ and $w(-x, -t)$ are also solutions to (1), which can be obtained by reversing time. These mean $\tilde{u}(x, t)$ and $\tilde{u}(x, t)$ solve (1). Moreover, through Proposition 4, we have $\lim_{t \to 0^+} \tilde{u}(x, t) = v(-x) = \lim_{t \to 0^+} \tilde{w}(x, t)$.
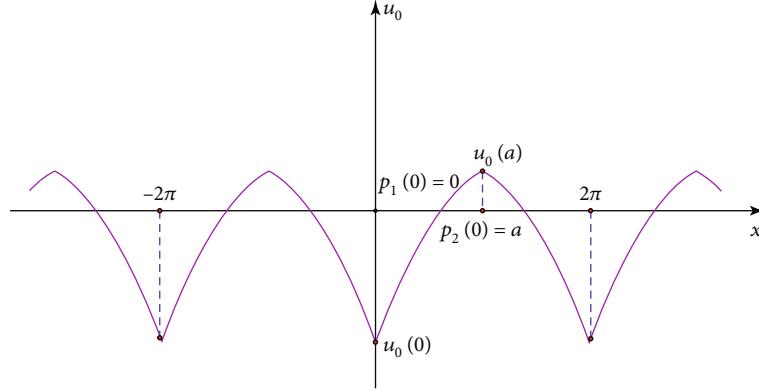
FIGURE 2: Initial profile $u_0(x)$.

Finally, we note that the initial data for these nonunique solutions can be made arbitrarily small. Since the new initial data is the collision function $v(x) = -\delta e^{-|x-q_\tau|}$, we have

$$\|v\|_{H^s}^2 = 4\mu\delta^2, \text{ where } \mu = \int_{\mathbb{R}} \left(1 + \xi^2\right)^{s-2} d\xi. \tag{46}$$

Therefore, for any $\varepsilon > 0$, we can find a δ to make $\|v\|_{H^s} < \varepsilon$. □

4. Nonuniqueness on the Circle

In this section, we use the ODE system (10) to prove the Theorem 1 on the circle. To do this, we take a 2-peakon solution of the form (9). From the first two items of the system (10), $p_1(t) = p_1(0)$ and $p_2(t) = p_2(0)$ are obvious. At the same time, we have $q_1{}' = q_2{}'$ if we take the symmetric initial data, $|p_1(0)| = |p_2(0)|$. The two peaks move at the same speed which means there is no collision. Therefore, we choose the same initial data as the line case

$$
\begin{aligned}
p_1(0) &= -(b + \delta), \quad q_1(0) = 0, \\
p_2 &= b, \quad\quad\quad\ q_2(0) = a,
\end{aligned}
\tag{47}
$$

with $0 < a < \pi$ and $b^2 + (b + \delta)^2 > (10\mathrm{ch}^2(\pi)/2\mathrm{ch}^2(\pi) + 3)b(b + \delta)$, where $b + \delta > b > 0$. The selection of these initial data is summarized in Figure 2.

According to (47), $p_1(t) = -(b + \delta)$ and $p_2(t) = b$ are obtained. We introduce the symbol q to represent the difference between the positions of the two peakons, in other words, $q \doteq q_2 - q_1$. It follows from the ODE system (10) that

$$
\begin{aligned}
q' &= \frac{8}{15}\left(p_2^4 - p_1^4\right)\mathrm{sh}^2(\pi)\left(\mathrm{ch}^2(\pi) + \frac{3}{2}\right)(1 + \alpha_2\mathrm{ch}(\pi - q)) \\
&= -\frac{8}{15}\alpha_1(1 + \alpha_2\mathrm{ch}(\pi - q)),
\end{aligned}
\tag{48}
$$

where

$$\alpha_1 = \left((b + \delta)^4 - b^4\right)\mathrm{sh}^2(\pi)\left(\mathrm{ch}^2(\pi) + \frac{3}{2}\right), \tag{49}$$

$$\alpha_2 = \frac{5p_1p_2}{p_1^2 + p_2^2} \cdot \frac{\mathrm{ch}(\pi)}{\mathrm{ch}^2(\pi) + 3/2} = \frac{-5b(b + \delta)}{b^2 + (b + \delta)^2} \cdot \frac{\mathrm{ch}(\pi)}{\mathrm{ch}^2(\pi) + 3/2}. \tag{50}$$

These are easy to get from (49) and (50) that $\alpha_1 > 0$ and $-1/\mathrm{ch}(\pi) < \alpha_2 < 0$. So, we have $1 + \alpha_2\mathrm{ch}(\pi - q) > 0$, which leads to $q' < 0$. Integrating (48), we calculate

$$g(q) - g(q_0) = \frac{4}{15}\alpha_1\alpha_2 t, \tag{51}$$

where

$$g(x) = \ln \frac{\sqrt{1 - \alpha_2^2} + 1 + \alpha_2 e^{\pi-x}}{\sqrt{1 - \alpha_2^2} - 1 - \alpha_2 e^{\pi-x}}. \tag{52}$$

Differentiating (52), we have

$$g'(x) = \frac{-2\alpha_2\sqrt{1 - \alpha_2^2}e^{\pi-x}}{\left(\sqrt{1 - \alpha_2^2} + 1 + \alpha_2 e^{\pi-x}\right)\left(\sqrt{1 - \alpha_2^2} - 1 - \alpha_2 e^{\pi-x}\right)}. \tag{53}$$

Since $-\sqrt{1 - \alpha_2^2} < 1 + \alpha_2 e^{\pi-x} < \sqrt{1 - \alpha_2^2}$, we find $g'(x) > 0$. It follows that $g(x)$ increases in $[0, \pi)$. Combined with $q_0 = q_2(0) - q_1(0) = a > 0$, we obtain a collision and a positive collision time when $q(t) = 0$. Using the symbol τ for the collision time, from (51), we find

$$\tau = \frac{15}{4\alpha_1\alpha_2}\left(\ln \frac{\sqrt{1 - \alpha_2^2} + 1 + \alpha_2 e^{\pi}}{\sqrt{1 - \alpha_2^2} - 1 - \alpha_2 e^{\pi}} - \ln \frac{\sqrt{1 - \alpha_2^2} + 1 + \alpha_2 e^{\pi-a}}{\sqrt{1 - \alpha_2^2} - 1 - \alpha_2 e^{\pi-a}}\right). \tag{54}$$

Applying $q_\tau \doteq \lim\limits_{t \longrightarrow \tau^-} q_1(t) = \lim\limits_{t \longrightarrow \tau^-} q_2(t)$ and $v(x) \doteq -\delta \cdot \mathrm{ch}$ $([x - q_\tau(t)]_p - \pi)$ to define the collision location q_τ and the collision function v, we get the following proposition.

Proposition 5. *The H^s limit of u as t approaches τ is v, or*

$$\lim_{t \longrightarrow \tau^-} \|u(t) - v\|_{H^s} = 0. \tag{55}$$

Proof. We compute the Fourier transform of u and v, which is denoted by

$$
\begin{aligned}
\widehat{u}(n, t) &= \mathrm{sh}(\pi)\left(\frac{2p_1 e^{-\mathrm{in}q_1}}{1 + n^2} + \frac{2p_2 e^{-\mathrm{in}q_2}}{1 + n^2} \right) \\
&= \mathrm{sh}(\pi)\left(\frac{-2(b+\delta)e^{-\mathrm{in}q_1}}{1 + n^2} + \frac{2b e^{-\mathrm{in}q_2}}{1 + n^2} \right),
\end{aligned}
\tag{56}
$$

$$\widehat{v}(n) = -\mathrm{sh}(\pi) \cdot \frac{2\delta e^{-\mathrm{in}q_\tau}}{1 + n^2}. \tag{57}$$

Combining (56) and (57), we have

$$
\begin{aligned}
\lim_{t \longrightarrow \tau^-} \|u(t) - v\|_{H^s}^2 &= \lim_{t \longrightarrow \tau^-} 4\mathrm{sh}^2(\pi) \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{s-2} \\
&\quad \cdot \left| b e^{-\mathrm{in}q_2} - (b+\delta)e^{-\mathrm{in}q_1} + \delta e^{-\mathrm{in}q_\tau} \right|^2.
\end{aligned}
\tag{58}
$$

Notice that the equation inside the absolute value can be scaled up to

$$
\begin{aligned}
& \left| b e^{-\mathrm{in}q_2} - (b+\delta)e^{-\mathrm{in}q_1} + \delta e^{-\mathrm{in}q_\tau} \right| \\
& \le (b+\delta)\left| e^{-\mathrm{in}q_1} \right| + b\left| e^{-\mathrm{in}q_2} \right| + \delta\left| e^{-\mathrm{in}q_\tau} \right| = 2(b+\delta).
\end{aligned}
\tag{59}
$$

Let $h(n) \doteq 4(b+\delta)^2 (1 + n^2)^{s-2}$. h is addable when $s < 3/2$ and h dominate the original integrand, which means that we can apply the dominated convergence theorem and put the limit inside the integral. So, we get

$$
\begin{aligned}
& \lim_{t \longrightarrow \tau^-} 4\mathrm{sh}^2(\pi) \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{s-2} \left| b e^{-\mathrm{in}q_2} - (b+\delta)e^{-\mathrm{in}q_1} + \delta e^{-\mathrm{in}q_\tau} \right|^2 \\
& = 4\mathrm{sh}^2(\pi) \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{s-2} \left| b e^{-\mathrm{in}q_2(\tau)} - (b+\delta)e^{-\mathrm{in}q_1(\tau)} + \delta e^{-\mathrm{in}q_\tau} \right|^2 = 0.
\end{aligned}
\tag{60}
$$

□

Proof of Theorem 1 (On the circle). In view of the 2-peakon solution we constructed, we need to construct a traveling wave solution $w(x, t)$ that satisfies

$$w(x, \tau) = u(x, \tau) = v(x). \tag{61}$$

Reviewing the system (10), we take the following data

$$
\begin{aligned}
p_1 &= -\delta, \quad q_1(0) = \beta, \\
p_2 &= 0, \quad\quad q_2(0) = 0.
\end{aligned}
\tag{62}
$$

The system is simplified and easy to be solved as

$$p_1(t) = -\delta, \quad q_1(t) = \left[\frac{8}{15}\mathrm{ch}^4(\pi) + \frac{4}{15}\mathrm{ch}^2(\pi) + \frac{1}{5} \right]\delta^4 t + \beta. \tag{63}$$

We introduce the symbol q_1^u to represent q_1 corresponding to u and q_1^w to represent q_1 corresponding to w. It follows from (61) that $q_1^u(\tau) = q_1^w(\tau)$. So, we get

$$
\begin{aligned}
\beta = {}& q_1^u(\tau) - \frac{2\delta^4}{\alpha_1\alpha_2}\left(\mathrm{ch}^4(\pi) + \frac{1}{2}\mathrm{ch}^2(\pi) + \frac{3}{8} \right) \\
& \cdot \left(\ln \frac{\sqrt{1 - \alpha_2^2} + 1 + \alpha_2 e^\pi}{\sqrt{1 - \alpha_2^2} - 1 - \alpha_2 e^\pi} - \ln \frac{\sqrt{1 - \alpha_2^2} + 1 + \alpha_2 e^{\pi-a}}{\sqrt{1 - \alpha_2^2} - 1 - \alpha_2 e^{\pi-a}} \right),
\end{aligned}
\tag{64}
$$

which makes the construction complete. In order to prove the nonuniqueness of the solution, we define two new solutions $\tilde{u}(x, t) \doteq u(-x, -t + \tau)$, $\tilde{w}(x, t) \doteq w(-x, -t + \tau)$. Since $u(x, t)$ and $w(x, t)$ are two solutions to (1), $u(-x, -t)$ and $w(-x, -t)$ are also solutions to (1), which can be obtained by reversing time. These mean $\tilde{u}(x, t)$ and $\tilde{u}(x, t)$ solve (1). In addition, similar to the line case, through Proposition 5, we have $\lim\limits_{t \longrightarrow 0^+} \tilde{u}(x, t) = v(-x) = \lim\limits_{t \longrightarrow 0^+} \tilde{w}(x, t)$. Finally, we note that the initial data for these nonunique solutions can be made arbitrarily small. Since the new initial data is the collision function $v(x) = -\delta \cdot \mathrm{ch}([x - q_\tau(t)]_p - \pi)$, we have

$$\|v\|_{H^s}^2 = 4\mu\delta^2\mathrm{sh}^2(\pi), \text{ where } \mu = \sum_{n\in\mathbb{Z}} \left(1 + n^2\right)^{s-2}. \tag{65}$$

Therefore, for any $\varepsilon > 0$, we can find a δ to make $\|v\|_{H^s} < \varepsilon$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors read and approved the final manuscript.

Acknowledgments

References

[1] R. Camassa and D. D. Holm, "An integrable shallow water equation with peaked solitons," *Physical Review Letters*, vol. 71, no. 11, pp. 1661–1664, 1993.

[2] M. Fisher and J. Schiff, "The Camassa Holm equation: conserved quantities and the initial value problem," *Physics Letters A*, vol. 259, no. 5, pp. 371–376, 1999.

[3] B. Fuchssteiner and A. S. Fokas, "Symplectic structures, their Backlund transformations and hereditary symmetries," *Physica D: Nonlinear Phenomena*, vol. 4, no. 1, pp. 47–66, 1981.

[4] M. S. Alber, R. Camassa, D. D. Holm, and J. E. Marsden, "The geometry of peaked solitons and billiard solutions of a class of integrable PDE's," *Letters in Mathematical Physics*, vol. 32, no. 2, pp. 137–151, 1994.

[5] C. S. Cao, D. D. Holm, and E. S. Titi, "Traveling wave solutions for a class of one-dimensional nonlinear shallow water wave models," *Journal of Dynamics and Differential Equations*, vol. 16, no. 1, pp. 167–178, 2004.

[6] R. Beals, D. H. Sattinger, and J. Szmigeilski, "Multi-peakons and a theorem of Stieltjes," *Inverse Problems*, vol. 15, no. 1, pp. L1–L4, 1999.

[7] R. Beals, D. H. Sattinger, and J. Szmigeilski, "Multipeakons and the classical moment problem," *Advances in Mathematics*, vol. 154, no. 2, pp. 229–257, 2000.

[8] A. Degasperis and M. Procesi, "Asymptotic integrability," *Symmetry and Perturbation Theory*, vol. 1, no. 1, pp. 23–37, 1999.

[9] A. Degasperis, D. D. Holm, and A. N. W. Hone, "A new integrable equation with peakon solutions," *Theoretical and Mathematical Physics*, vol. 133, no. 2, pp. 1463–1474, 2002.

[10] H. R. Dullin, G. A. Gottwald, and D. D. Holm, "Camassa-Holm, Korteweg–de Vries-5 and other asymptotically equivalent equations for shallow water waves," *Fluid Dynamics Research*, vol. 33, no. 1-2, pp. 73–95, 2003.

[11] H. Lundmark and J. Szmigeilski, "Multi-peakon solutions of the Degasperis-Procesi equation," *Inverse Problems*, vol. 19, no. 6, pp. 1241–1245, 2003.

[12] H. Lundmark and J. Szmigeilski, "Degasperis-Procesi peakons and the dis- crete cubic string," *International Mathematics Research Papers*, vol. 2005, no. 2, pp. 53–116, 2005.

[13] J. Szmigeilski and L. Zhou, "Peakon-antipeakon interactions in the Degasperis-Procesi equation," *Contemporary Mathematics*, vol. 593, pp. 83–107, 2013.

[14] A. Fokas, "The Korteweg-de Vries equation and beyond," *Acta Applicandae Mathematicae*, vol. 39, no. 1-3, pp. 295–305, 1995.

[15] P. J. Olver and P. Rosenau, "Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support," *Physical Review E. Statistical, Nonlinear, and Soft Matter Physics*, vol. 53, no. 2, pp. 1900–1906, 1996.

[16] A. S. Fokas, P. J. Olver, and P. Rosenau, "A plethora of integrable bi- Hamiltonian equations. In: algebraic aspects of Integrable systems," *Progress in Nonlinear Differential Equations and their Applications*, vol. 26, pp. 93–101, 1997.

[17] Z. Qiao and X. Q. Li, "An integrable equation with non-smooth solitons," *Theoretical and Mathematical Physics*, vol. 167, no. 2, pp. 584–589, 2011.

[18] Z. Qiao, "A new integrable equation with cuspons and W/M-shape-peaks solitons," *Journal of Mathematical Physics*, vol. 47, no. 11, pp. 1661–1664, 2006.

[19] G. Gui, Y. Liu, P. J. Olver, and C. Qu, "Wave-breaking and peakons for a modified Camassa-Holm equation," *Communications in Mathematical Physics*, vol. 319, no. 3, pp. 731–759, 2013.

[20] V. Novikov, "Generalizations of the Camassa-Holm equation," *Journal of Physics. A. Mathematical and Theoretical*, vol. 42, no. 34, article 342002, 2009.

[21] A. N. W. Hone and J. P. Wang, "Integrable peakon equations with cubic nonlinearity," *Journal of Physics. A. Mathematical and Theoretical*, vol. 41, no. 37, article 372002, 2008.

[22] A. N. W. Hone, H. Lundmark, and J. Szmigielski, "Explicit multipeakon solutions of Novikovs cubically nonlinear integrable Camassa-Holm type equation," *Dynamics of Partial Differential Equations*, vol. 6, no. 3, pp. 253–289, 2009.

[23] D. D. Holm and A. N. W. Hone, "A class of equations with peakon and pulson solutions (with an Appendix by Harry Braden and John Byatt-Smith)," *Journal of Nonlinear Mathematical Physics*, vol. 12, Supplement 1, pp. 380–394, 2005.

[24] Y. Mi and C. Mu, "On the Cauchy problem for the modified Novikov equation with peakon solutions," *Journal of Differential Equations*, vol. 254, no. 3, pp. 961–982, 2013.

[25] A. A. Himonas and D. Mantzavinos, "An ab-family of equations with peakon traveling waves," *Proceedings of the American Mathematical Society*, vol. 144, no. 9, pp. 3797–3811, 2016.

[26] Y. A. Li and P. J. Olver, "Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation," *Journal of Differential Equations*, vol. 162, no. 1, pp. 27–63, 2000.

[27] P. Byers, "Existence time for the Camassa-Holm equation and the critical Sobolev index," *Indiana University Mathematics Journal*, vol. 55, no. 3, pp. 941–954, 2006.

[28] A. A. Himonas, C. Holliman, and K. Grayshan, "Norm inflation and ill-posedness for the Degasperis-Procesi equation," *Communications in Partial Differential Equations*, vol. 39, no. 12, pp. 2198–2215, 2014.

[29] A. A. Himonas and C. Holliman, "The Cauchy problem for the Novikov equation," *Nonlinearity*, vol. 25, no. 2, pp. 449–479, 2012.

[30] A. A. Himonas, C. Holliman, and C. Kenig, "Construction of 2-peakon solutions and ill-posedness for the Novikov equation," *SIAM Journal on Mathematical Analysis*, vol. 50, no. 3, pp. 2968–3006, 2018.

[31] Z. Guo, L. Molinet, and Z. Yin, "Ill-posedness of the Camassa-Holm and related equations in the critical space," *Journal of Differential Equations*, vol. 266, no. 2-3, pp. 1698–1707, 2019.

[32] A. A. Himonas and D. Mantzavinos, "The Cauchy problem for the Fokas-Olver-Rosenau-Qiao equation," *Nonlinear Analysis*, vol. 95, pp. 499–529, 2014.

[33] A. A. Himonas and C. Holliman, "Non-uniqueness for the Fokas-Olver-Rosenau-Qiao equation," *Journal of Mathematical Analysis and Applica- tions*, vol. 470, no. 1, pp. 647–658, 2019.

[34] J. Holmes and R. Puri, "Non-uniqueness for the ab-family of equations," *Journal of Mathematical Analysis and Applications*, vol. 493, no. 2, article 124563, 2021.

[35] A. A. Himonas, K. Grayshan, and C. Holliman, "Ill-posedness for the b- family of equations," *Journal of Nonlinear Science*, vol. 26, no. 5, pp. 1175–1190, 2016.

[36] E. Novruzov, "Construction of peakon-antipeakon solutions and ill-posedness for the b-family of equations," *Journal of Differential Equations*, vol. 272, pp. 544–559, 2021.

[37] S. C. Anco and E. Recio, "A general family of multi-peakon equations and their properties," *Journal of Physics. A. Mathematical and Theoretical*, vol. 52, no. 12, pp. 125203–125236, 2019.

[38] E. Recio and S. C. Anco, "Conserved norms and related conservation laws for multi-peakon equations," *Journal of Physics. A. Mathematical and Theoretical*, vol. 51, no. 6, pp. 065203–065219, 2018.