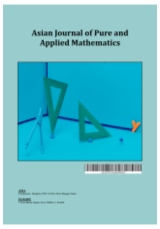

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Numerical Solution of Boundary Value Problems by Bernoulli Wavelet Based Galerkin Method

L. M. Angadi [a*]

[a] *Department of Mathematics, Shri Siddeshwar Government First Grade College & P. G. Studies Centre, Nargund – 582207, India.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, I introduced a numerical approach to obtain the solution of boundary value problems (BVPs) utilizing the Bernoulli wavelet-based Galerkin method (BWGM). In this context, employed weight functions represented as Bernoulli wavelets, which serve as the basis elements enabling us to derive the numerical solutions for the BVPs. The numerical solutions obtained through this method are contrasted with those from established methods and the exact solutions. Several BVPs are selected to illustrate the efficiency and relevance of the proposed methodology.

Keywords: Boundary value problems; Bernoulli wavelets; Galerkin method.

**Corresponding author: E-mail: angadi.lm@gmail.com;*

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1 Introduction

Boundary value problems (BVPs) are frequently observed across multiple fields of engineering and science, such as gas dynamics, nuclear physics, atomic structures, and chemical reactions. Often, exact solutions to these problems cannot be obtained through analytical techniques. In review, the numerical approximation of solutions to BVPs serves as a crucial mechanism across various scientific and engineering fields, facilitating the assessment and analysis of complex systems that are not readily solvable through analytical methods. Due to this, the numerical methods are very crucial. As a result, numerical methods are of significant importance.

Consequently, the development of numerical methods to derive approximate solutions is of paramount importance.

Recently, several numerical techniques have been employed to solve differential equations, such as the Galerkin method using Boubaker wavelet [1], the Fibonacci wavelet collocation method [2], and the wavelet-based Galerkin method [3], etc.

The area of wavelets has garnered considerable attention owing to their robust mathematical properties and extensive applicability in a variety of compelling physical issues. Recently, there has been a significant increase in the interest surrounding wavelet functions among researchers engaged in both theoretical and practical domains.

Expect progress in numerical techniques that employ wavelet bases to get enhanced spatial and spectral resolutions. A key principle in approximation theory is the representation of a smooth function as a series expansion through orthogonal polynomials. This approach underpins spectral methods used to solve differential equations with functional arguments. The investigation of wavelet function bases is being explored as an alternative to traditional piecewise polynomial trial functions in the finite element analysis of differential equations. The Galerkin method is extensively recognized in the field of applied mathematics for its effectiveness and ease of use [4-5].

The wavelet-Galerkin method offers significant advantages over both finite difference and finite element methods, which contributes to its widespread application in various scientific and engineering fields. This wavelet approach presents a robust alternative to the finite element method, particularly in the numerical resolution of differential equations, especially in the context of boundary value problems.

This study, I introduced the Bernoulli wavelet-based Galerkin (BWGM) method for addressing boundary value problems (BVPs) numerically. This approach involves representing the solution through Bernoulli wavelets characterized by unknown coefficients. Utilizing the characteristics of Bernoulli wavelets alongside the Galerkin method, I am able for obtaining the unknown coefficients, ultimately leading to the numerical solution for the BVPs.

Organization of the paper is presented as: Section 2 presents Bernoulli wavelets and function approximation. Section 3 focuses on the Bernoulli wavelet-based Galerkin method (BWGM) for addressing boundary value problems (BVPs). Section 4 contains the numerical experiments conducted. Lastly, section 5 provides a discussion of the conclusions drawn from the proposed work.

2 Bernoulli Wavelets and Function Approximation

2.1 Bernoulli wavelets

Bernoulli wavelets are defined as follows [6]:

The Bernoulli wavelets $\psi_{n,m}(x) = \psi(k, n, m, x)$ include four arguments: $n = 1, 2, ..., 2^{k-1}, k$ is assumed to be a positive integer, m is the order of Bernoulli polynomials and x is to be a normalized time. They are defined in the interval $[0,1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{B}_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

with

$$\tilde{B}_m(x) = \begin{cases} 1, & m = 0 \\ \dfrac{1}{\sqrt{\left(\frac{(-1)^{m-1}(m!)^2}{(2m)!}\right)\alpha_{2m}}}, & m > 0 \end{cases} \tag{2.2}$$

Where $m = 0,1,2,...,M-1$ and $n = 1,2,...,2^{k-1}$. The coefficient $\dfrac{1}{\sqrt{\left(\frac{(-1)^{m-1}(m!)^2}{(2m)!}\right)\alpha_{2m}}}$ is for normality, the dilation parameter is $a = 2^{-(k-1)}$ and the translation parameter is $b = (n-1)2^{-(k-1)}$. Here, $B_m(x) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-1} x^i$ are the Bernoulli polynomials of order m and α_i, $i = 0, 1, 2, ..., m$ are Bernoulli numbers. This sequence consists of signed rational numbers that emerge from the series expansion of trigonometric functions and can be defiened by $\frac{x}{e^x - 1} = \sum_{i=0}^{m} \alpha_i \frac{x^i}{i!}$.

Few Bernoulli numbers are $\alpha_0 = 1$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = \frac{1}{6}$, ... and the first few Bernoulli polynomials are $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, ...

For illustration, if $k = 1$ and $M = 3$, we obtain the Bernoulli wavelet bases are:

$\psi_{1,0}(x) = 1$, $\psi_{1,1}(x) = \sqrt{3}\,(2x - 1)$, $\psi_{1,2}(x) = \sqrt{5}\left(6x^2 - 6x + 1\right)$ and so on.

Function approximation

Suppose $y(x) \in L^2[0, 1)$ and the expression is represented in terms of Bernoulli wavelets as:

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \tag{2.3}$$

Truncating the aforementioned infinite series, we get

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \tag{2.4}$$

3 Method of Solution

Take the BVP of the type,

$$y'' + \alpha y' + \beta y = f(x) \tag{3.1}$$

Boundary conditions are $y(0) = a, \quad y(1) = b$ (3.2)

Where α , β are constants and $f(x)$ to be a continuous function.

Rewrite the Eq. (3.1) i.e. $R(x) = y'' + \alpha y' + \beta y - f(x)$ (3.3)

Here, $R(x)$ is the residual of the Eq. (3.1). For the exact solution if $R(x) = 0$ and $y(x)$ will met the boundary conditions.

The solution of Eq. (3.1) in trail series is defined in the interval [0, 1) and it can be expressed as a modified form of Bernoulli wavelets. This expansion adheres to the specified boundary conditions and incorporates an unknown parameter as outlined.

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \tag{3.4}$$

where $c_{n,m}$'s are the unknown coefficients and are to be determined.

The precision of the solution is enhanced by selecting Bernoulli wavelet polynomials of a higher degree. The process involves differentiate Eq. (3.4) with respect to the specified variable x twice, and then substitute the corresponding values of y , y' , y'' in Eq. (3.3). To determine the desired outcome, we select weight functions as the assumed basis elements and perform integration over the boundary values, ensuring that the residual is equal to zero [7].

i.e. $$\int_0^1 \psi_{1,m}(x) \, R(x) \, dx = 0, \; m = 0, 1, 2, ...$$

Consequently, I obtain a linear system of algebraic equations. Upon solved this system, I determined the unknown coefficients. These unknowns are then substituted into the trial solution, specifically Eq. (3.4), which yields the numerical solution of Eq. (3.1).

To assess the accuracy of the Bernoulli wavelet Galerkin method (BWGM) to the test problems, we employ the error metric known as the maximum absolute error. The calculation of the maximum absolute error will be conducted as follows.

$$E_{\max} = \max \left| y(x)_{exact} - y(x)_{approx} \right|.$$

Where $y(x)_{exact}$ and $y(x)_{approx}$ are exact and approximate solutions respectively.

4 Numerical Implementation

Problem 4.1 First, take the BVP

$$y'' - y' = -\left(e^{x-1} + 1\right), \quad 0 \le x \le 1 \tag{4.1}$$

$$\text{Boundary conditions: } y(0) = 0, \quad y(1) = 0 \tag{4.2}$$

The execution of the Eq. (4.1) as per the method discussed in section 3 is as follows:
Residual of Eq. (4.1) is i.e.

$$R(x) = y'' - y' + \left(e^{x-1} + 1\right) \tag{4.3}$$

At this point, the selection of the weight function $w(x) = x(1-x)$ for Bernoulli wavelet bases must satisfy the boundary conditions (Eq. (4.2)), i.e. $\psi(x) = w(x) \times \psi(x)$

$$\psi_{1,0}(x) = \psi_{1,0}(x) \times x(1-x) = x(1-x)$$
$$\psi_{1,1}(x) = \psi_{1,1}(x) \times x(1-x) = \sqrt{3}\,(2x-1)\,x(1-x)$$
$$\psi_{1,2}(x) = \psi_{1,2}(x) \times x(1-x) = \sqrt{5}\left(6x^2 - 6x + 1\right)x(1-x)$$

Assume the trail solution for Eq. (4.1) when $k = 1$ and $m = 2$ is given as

$$y(x) = c_{1,0}\,\psi_{1,0}(x) + c_{1,1}\,\psi_{1,1}(x) + c_{1,2}\,\psi_{1,2}(x) \tag{4.4}$$

Therefore, the Eq. (4.4) become

$$y(x) = c_{1,0}\left\{x(1-x)\right\} + c_{1,1}\left\{\sqrt{3}\,(2x-1)\,x(1-x)\right\} + c_{1,2}\left\{\sqrt{5}\left(6x^2 - 6x + 1\right)x(1-x)\right\} \tag{4.5}$$

Differentiate Eq. (4.5) w.r.t. to the variable x twice and substitute the values y', y'' in Eq. (4.3), and obtained the residual for Eq. (4.1). The "weight functions" are as same as the wavelet bases.

Subsequently, utilizing the weighted Galerkin method, we examine the following:

$$\int_0^1 \psi_{1,j}(x)\, R(x)\, dx = 0, \quad j = 0, 1, 2 \tag{4.6}$$

For $j = 0, 1, 2$ in Eq. (4.6),

i.e.
$$\left.\begin{aligned} \int_0^1 \psi_{1,0}(x)\, R(x)\, dx &= 0 \\ \int_0^1 \psi_{1,1}(x)\, R(x)\, dx &= 0 \\ \int_0^1 \psi_{1,2}(x)\, R(x)\, dx &= 0 \end{aligned}\right\} \tag{4.7}$$

A system of algebraic equations with unknown coefficients are obtained using Eq. (4.7) i.e. $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. By applying the Gauss elimination method to solve this system, we determine the values of $c_{1,0} = 0.7967$, $c_{1,1} = 0.1049$ and $c_{1,2} = 0.0087$. Substitute these values in Eq. (4.5) yields the numerical solution. The obtained numerical solution and the corresponding absolute errors are displayed in Table 1, the evaluation of the numerical solution in relation to the exact solution of Equation (4.1). The evaluation of the numerical solution in relation to the exact solution of Eq. (4.1) $y(x) = x\left(1 - e^{x-1}\right)$ [8] is illustrated in Fig. 1.

Table 1. Comparison of the numerical solution and the absolute error in relation to the exact solution for Problem 4.1

x	Numerical solution			Exact solution	Absolute error		
	FDM	**Ref [8]**	**BWGM**		**FDM**	**Ref [8]**	**BWGM**
0.1	0.061948	0.059383	0.059427	0.059343	2.61e-03	4.00e-05	8.40e-05
0.2	0.115151	0.110234	0.110154	0.110134	5.02e-03	1.00e-04	2.00e-05
0.3	0.158162	0.151200	0.150983	0.151024	7.14e-03	1.76e-04	4.10e-05
0.4	0.189323	0.180617	0.180433	0.180475	8.85e-03	1.42e-04	4.20e-05
0.5	0.206737	0.196983	0.196743	0.196735	1.00e-02	2.48e-04	8.00e-06
0.6	0.208235	0.198083	0.197875	0.197808	1.04e-02	2.75e-04	6.70e-05
0.7	0.191342	0.181655	0.181507	0.181427	9.92e-03	2.28e-04	8.00e-05
0.8	0.153228	0.145200	0.145039	0.145015	8.21e-03	1.85e-04	2.40e-05
0.9	0.090672	0.085710	0.085590	0.085646	5.03e-03	6.40e-05	5.60e-05

FDM: Finite difference method **BWGM:** Bernoulli wavelet Galerkin method

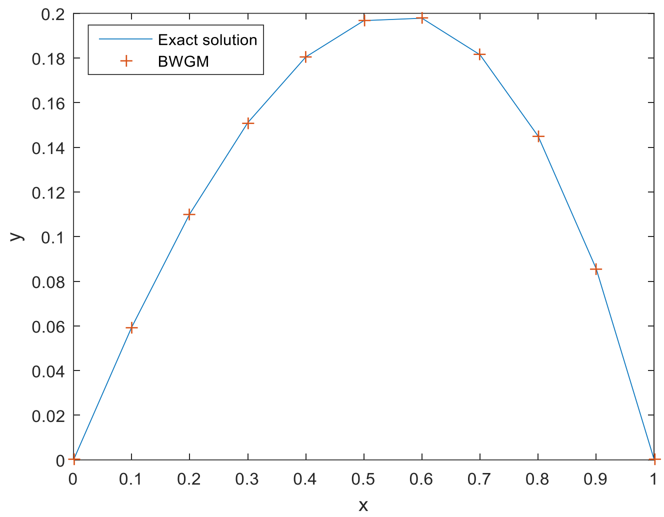

Fig. 1. Comparison of the numerical solution with the exact solution for Problem 4.1.

Problem 4.2 Next, take another BVP

$$y'' - \pi^2 y = -2\pi^2 \sin(\pi x), \quad 0 \le x \le 1 \tag{4.8}$$

Boundary conditions: $y(0) = 0$, $y(1) = 0$ (4.9)

As explained in Section 3 and the problem 4.1, obtained the values of unknown coefficients i.e. $c_{1,0} = 3.702$, $c_{1,1} = 0.0$ and $c_{1,2} = -0.2634$. Put these values in Eq. (4.5) and arrived the numerical solution. The comparison between the numerical solution and the absolute errors is illustrated in Table 2, while the numerical solution is contrasted with the exact solution of Eq. (4.8) $y(x) = \sin(\pi x)$ [3] in Fig. 2.

Table 2. The numerical solution and the absolute error are compared with the exact solution for the problem 4.2

x	Numerical solution			Exact solution	Absolute error		
	FDM	Ref [3]	BWGM		FDM	Ref [3]	BWGM
0.1	0.310289	0.308754	0.308796	0.309016	1.27e-03	2.60e-04	2.20e-04
0.2	0.590204	0.588509	0.588551	0.588772	1.43e-03	2.60e-04	2.20e-04
0.3	0.812347	0.809554	0.809538	0.809016	3.33e-03	5.40e-04	5.30e-04
0.4	0.954971	0.950670	0.950676	0.951056	3.92e-03	3.90e-04	3.80e-04
0.5	1.004126	0.999123	0.999123	1.000000	4.13e-03	8.80e-04	8.80e-04
0.6	0.954971	0.950670	0.950676	0.951056	3.92e-03	3.90e-04	3.80e-04
0.7	0.812347	0.809554	0.809538	0.809016	3.33e-03	5.40e-04	5.30e-04
0.8	0.590204	0.588509	0.588486	0.587785	2.42e-03	7.20e-04	7.00e-04
0.9	0.310289	0.308754	0.308796	0.309016	1.27e-03	2.60e-04	2.20e-04

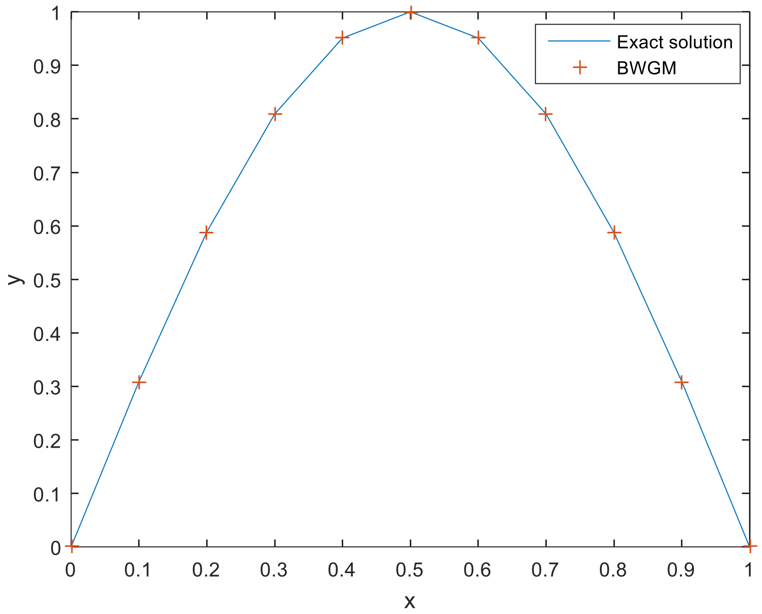

Fig. 2. A comparison between the numerical solution and the exact solution for Problem 4.2

Problem 4.3 Finally, another BVP,

$$y'' + y = x^2, \quad 0 \le x \le 1 \tag{4.10}$$

With boundary conditions: $y(0) = 0, \quad y(1) = 0$ (4.11)

As explained in Section 3 and the problem 4.1, obtained the values of unknown coefficients i.e. $c_{1,0} = -0.1689$, $c_{1,1} = -0.0493$ and $c_{1,2} = -0.0052$. Substitute the obtained values in Eq. (4.5), and get the numerical solution. The comparison between the numerical solution and the absolute errors is illustrated in tables 3(a) and 3(b), while the numerical solution is contrasted with the exact solution of Eq. (4.10) $y(x) = \dfrac{\sin(x) + 2\sin(1-x)}{\sin(1)} + x^2 - 2$ [9] as shown in Fig. 3.

Table 3(a). A comparison of the numerical solution and the absolute error in relation to the exact solution for problem 4.3

x	Numerical solution		Exact solution	Absolute error	
	FDM	BWGM		FDM	BWGM
0.1	-0.009628	-0.0095343	-0.009555	7.30e-05	2.10e-05
0.2	-0.019027	-0.0189010	-0.018897	1.30e-04	4.00e-06
0.3	-0.027804	-0.0276614	-0.027635	1.69e-04	2.60e-05
0.4	-0.035371	-0.0352094	-0.035180	1.91e-04	2.90e-05
0.5	-0.040954	-0.0407716	-0.040759	1.95e-04	1.30e-05
0.6	-0.043600	-0.0434069	-0.043416	1.84e-04	9.10e-06
0.7	-0.042180	-0.0420069	-0.042025	1.55e-04	1.80e-05
0.8	-0.035418	-0.0352959	-0.035302	1.16e-04	6.10e-06
0.9	-0.021878	-0.0218305	-0.021815	6.30e-05	1.50e-05

Table 3(b). Comparison between the numerical solution and absolute error with the exact solution for the problem 4.3

x	Numerical solution		Exact solution	Absolute error	
	Ref [9]	**BWGM**		**Ref [9]**	**BWGM**
0.125	-0.0121	-0.0119059	-0.0119	2.00e-04	5.90e-06
0.375	-0.0340	-0.0334755	-0.0334	6.00e-04	7.50e-05
0.625	-0.0440	-0.0434821	-0.0435	5.00e-04	1.80e-05
0.875	-0.0261	-0.0259153	-0.0259	2.00e-04	1.50e-05

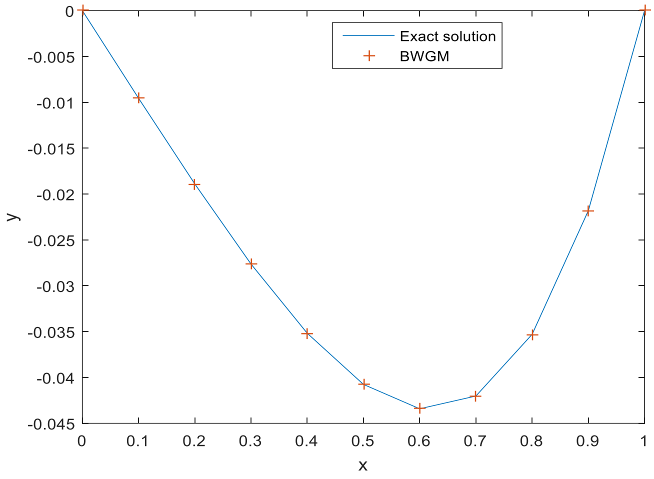

Fig. 3. A comparison between the numerical solution and the exact solution for Problem 4.3

5 Conclusions

This study presented the Bernoulli wavelet-based Galerkin approach for solving boundary value problems numerically. The advancement of new research in numerical analysis is significantly enhanced by this, proving advantageous for emerging researchers. The method introduced has been applied to some examples, yielding results that are notably satisfactory when compared to other established numerical methods. From the tables and figures presented above, I noted that:

- The numerical solutions derived from the proposed method demonstrate superior performance compared to the finite difference method (FDM) and some existing methods.
- The absolute error produced by this method is significantly lower in comparison to the Finite Difference Method (FDM) and some existing methods.

Consequently, the Galerkin method utilizing Bernoulli wavelets proves to be highly efficient in solving BVPs.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Angadi LM, Numerical solution of boundary value problems by Galerkin method using Boubaker wavelet, TWMS Journal of Applied and Engineering Mathematics. 2023;(1):145 – 153.

[2] Manohara G, Kumbinarasaiah S. An innovative Fibonacci wavelet collocation method for the numerical approximation of Emden-Fowler equations. 2024;201:347-369.

[3] Shiralashetti SC, Angadi LM, Kumbinarasaiah S. Wavelet based Galerkin Method for the Numerical Solution of One Dimensional Partial Differential Equations, International Research Journal of Engineering and Technology. 2019;6(7):2886-2896.

[4] Amaratunga K, William JR. Wavelet-Galerkin Solutions for One dimensional Partial Differential Equations, International Journal for Numerical Methods in Engineering. 1994;37:2703-2716. DOI:10.1002/NME.1620371602

[5] Mosevic JW. Identifying Differential Equations by Galerkin's Method, Mathematics of Computation. 1977;31:139-147.

[6] Kumbinarasaiah S, Manohara G, Hariharan G. Bernoulli wavelets functional matrix technique for a system of nonlinear singular Lane Emden equations. 2023;204:133-165. DOI: 10.1016/matcom.2022.07.024

[7] Cicelia JE, Solution of Weighted Residual Problems by using Galerkin's Method, Indian Journal of Science and Technology. 2014;7(3):52–54. DOI: 10.17485/ijst/2014/v7sp3.3

[8] Lot T, Mahdiani K. Numerical solution of boundary value problem by using wavelet Galerkin method, Mathematical Science. 2007;1:07–18.

[9] Arora S, Brar YS, Kumar S. Haar Wavelet Matrices for the Numerical Solutions of Differential Equations, International Journal of Computer Applications. 2014;97(18). DOI: 10.5120/17108-7759

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