
**British Journal of Mathematics & Computer Science**
**4(13): 1786-1796, 2014**

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# Closed form Solution to Nonlinear Equilibria and Oscillations of a Damped Eccentric Rotor

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***Original Research Paper***

*Received: 26 July 2013*
*Accepted: 14 April 2014*
*Published: 02 May 2014*

## Abstract

We study an eccentric and elastically damped rotor from both a statical and dynamical point of view. The system, whose genesis is in the re-loading mechanism of an automatic watch, behaves like a generalized physical pendulum with the addition of eccentricity and damping. The static analysis is performed by settling the statical equilibria and defining their nature, whose effective computation can be done numerically. The dynamical analysis leads to a nonlinear differential initial-value problem whose integration is carried out by means of Jacobi elliptic functions. It reveals that, starting from both positional and kinetic zero initial conditions, only periodical motions, see formulae (4.8) or (4.13), are allowed and all confined inside a potential well. Closed form expressions of the oscillation period have been obtained through complete elliptic integrals of the first kind. In such a way a further treatment is added to the non-rich collection of 1-D nonlinear oscillators suitable of closed form integration.

*Keywords: Single degree of freedom; nonlinear oscillations; rigid body; elliptic functions*
2010 Mathematics Subject Classification: 33E05; 34C15

## 1 Introduction

Most physical, biological and economic systems are inherently nonlinear so that they lead to nonlinear mathematical problems e.g. integral and/or differential ordinary equations. In such a way, many systems cannot be solved through the existing closed-form methods. A branch of applied research is then looking for exact solutions, by means of either the special functions of Mathematical Physics, iterative approach, perturbative methods, or by harmonic balance. One can see all this in recent papers like [1], [2], [3] concerning various topics on classical Mechanics. Specifically on vibrations: [4], [5], [6]. On the analytical solutions, we quote the point of view of A. A. [7]:

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The importance of analytical solutions for the equations characterizing physical systems is well known. While iterative solutions are useful, once a governing equation is obtained, a closed form solution makes the analysis far more elaborate and easier. The behavior of the solution is at once clear, when expressed in closed form in terms of known functions rather than generating extensive tables at specific values. Further, since the functional values can be pre-calculated, it makes repetitions of such calculations for specific cases redundant, thereby, saving a considerable amount of computing time.

The nonlinear pendulum has been a source of several researches not only for those generalizations which can be done, but also on its own. As a matter of fact its well known solution via elliptic integrals and Jacobi functions provides a powerful benchmark for testing some approximate methods to be next used on more difficult problems where an exact solution is not available. The sequence of articles by Belendez and alii dedicated to the nonlinear pendulum goes along this path in order to provide approximate expressions to its period [8], to analyze the quintic Duffing oscillator [9], to face the problem by linearization [10], or by cubication [11] or through the homotopy perturbation [12], or asymptotically [13]. In [14] and [15] the Laplace transform of the period is obtained of a one-dimensional conservative system whose potential has a unique minimum. The author makes use of the complex analysis in order to get the exact form of the period as a function of energy.

# 2 Our Mechanical System

We introduce our nonlinear system. A mechanical watch is notoriously powered by an internal spiral mainspring which turns the gears that move both hands. The spring loses energy as the watch runs: so that in a manual watch the spring must be wound by turning a small knob. A flywheel is a mechanical rotor for equalizing the speed of a shaft: we don't refer to such a type, but to its oscillating variant which operates the re-loading mechanism of an automatic watch. It contains an eccentric mass (the rotor), which turns on a pivot. The weight unbalance due to the normal movements of the user's arm causes the rotor to make angular oscillations on its staff, which is attached to a ratcheted winding mechanism and then loads the spring inside the watch's barrel.

The present article is based on a study on a an eccentrically pivoted and elastically damped rotor we came across during a consultancy about the applied mechanics: such a single degree of freedom system generated two different mathematical problems we approached in closed form, not a common habit in mechanical engineering. After the statical equilibria, we passed to the system dynamics: assuming two zero initial conditions we solve the Lagrange motion equation, discovering some parametrically different time behaviors of the rotor, which required the employ of elliptic functions.

This paper belongs to the research line applying the special functions to the nonlinear systems. In some our papers, as in [16] or in [17], the mechanical aspects have been omitted by us completely in order to focus on the mathematical ones; or just alluded, as in [18] when dealing with a 2-D system.

In this paper, on the contrary, some room will be devoted to the mechanical peculiarities of a 1-D not so simple system, as we are going to describe,of course in our model the rotor actions to all the surrounding are not included.

On a vertical plane, a rotor consisting of a homogeneous ring of mass $m$, radius $r$ and center $C$ is pivoted without friction at the origin $O$ of an orthogonal cartesian frame $Oxy$ with a hinge allowing only rotations around $O$. The $OS$ diameter of the ring is formed by a rigid homogeneous rod of $m$ mass too, and welded to the ring. At the $D$ point, see Figure 1, so that $O\widehat{C}D = \frac{\pi}{2}$ a block of mass $m$ is welded to the ring.

On the planar rigid body the weights of ring, block and rod are acting; furthermore an elastic force is exerted at B, so that $D\widehat{C}B = \frac{\pi}{2}$, and whose center is the projection $B'$ of $B$ on the axis $Oy$. All these are conservative forces and let $\theta$ be the single degree of freedom. The $Ox$ axis is

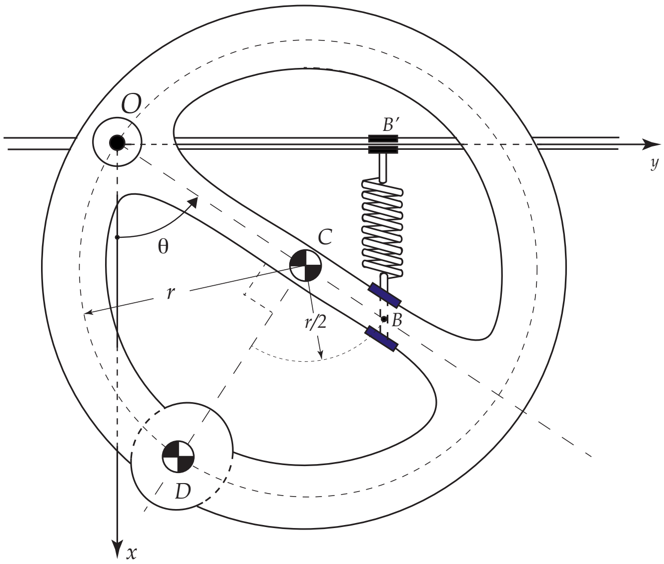


Figure 1: The eccentric damped rotor: circular ring, rod, block and spring.

downward and the unit vectors codirectional with the $x$, $y$ axes (sometimes referred to as versors of the coordinates) are marked as $\vec{i}$ and $\vec{j}$.

# 3 Statical Equilibria Search and Their Classification

Before the static analysis, let us locate the application points of all the loads:

$$\overrightarrow{OC} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j};\ \overrightarrow{OB} = \frac{3r}{2}\cos\theta\,\vec{i} + \frac{3r}{2}\sin\theta\,\vec{j};\ \overrightarrow{OB'} = \frac{3r}{2}\sin\theta\,\vec{j}.$$

So that

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = r(\cos\theta\ + \sin\theta)\,\vec{i} + r(\sin\theta - \cos\theta)\,\vec{j}.$$

The acting forces are:

- weight of each of three elements: $mg\,\vec{i}$
- elastic force $h\overrightarrow{BB'} = -h\frac{3r}{2}\cos\theta\,\vec{i}$
- constraint reaction from the O-hinge: $\overrightarrow{\Phi_O} = \Phi_{Ox}\,\vec{i} + \Phi_{Oy}\,\vec{j}$

Then the potential of all the forces on the system is:

$$\begin{aligned} U(\theta) &= U_{weight-ring} + U_{weight-rod} + U_{weight-block} + U_{elastic-force} \\ &= -mgx_C - mgx_C - mgx_D + \frac{h}{2}\left(\frac{3r}{2}\cos\theta\right)^2, \end{aligned}$$

or:

$$U(\theta) = -3mgr\cos\theta - mgr\sin\theta + \frac{9hr^2}{8}\cos^2\theta, \tag{3.1}$$

so that:

$$\frac{\mathrm{d}U}{\mathrm{d}\theta} = 3mgr\sin\theta - mgr\cos\theta - \frac{9}{4}hr^2\cos\theta\sin\theta. \tag{3.2}$$

The potential must be stationary: so all the equilibria can be detected as solutions of the equation $\frac{\mathrm{d}U}{\mathrm{d}\theta} = 0$, namely:

$$u(\theta) = 3\sin\theta - \cos\theta - p\sin\theta\cos\theta = 0, \quad 0 < \theta < 2\pi \tag{3.3}$$

where it has been put: $\frac{9hr}{4mg} = p > 0$. We are interested in defining, within the range of the $p$ positive parameter, the number and nature of the roots of function $u(\theta)$, equation (3.3). The answer comes from studying the cross with horizontal straight lines (in the half-plane of positive ordinates) of the line defined by:

$$f(\theta) = \frac{3\sin\theta - \cos\theta}{\sin\theta\cos\theta}, \quad 0 < \theta < 2\pi$$

Now, being

$$f'(\theta) = \frac{\cos^3\theta + 3\sin^3\theta}{\sin^2\theta\cos^2\theta}$$

we see that the point

$$\phi_0 = 2\pi - \arctan\frac{1}{\sqrt[3]{3}}, \quad p_0 = f(\phi_0) = \frac{\left(3+\sqrt[3]{3}\right)^{3/2}}{\sqrt{3}}$$

is a relative minimum, see Figure 2 below.

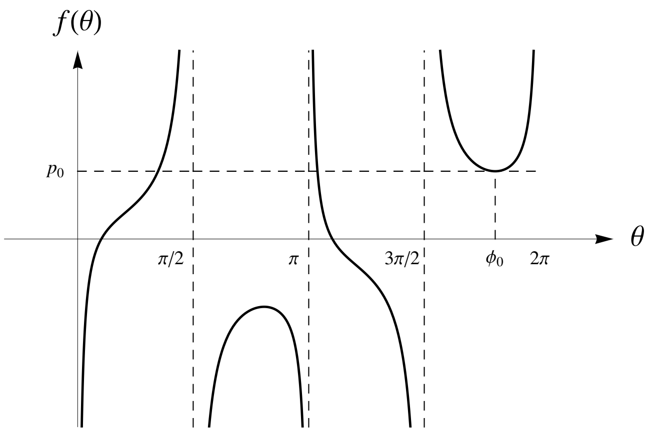


Figure 2: The static case.

Equation (3.3) has:

- two solutions for $0 < p < p_0$;
- three solutions for $p = p_0$ whose one is of multiplicity 2;
- four solutions for $p > p_0$.

Each of these solutions mean a static equilibrium point: after such analysis on the $f(\theta)$ roots, we will provide their classification. A relative maximum will mean a stable equilibrium position; and so on.

For each value of $p > 0$ we note, see eq. (3.3), that $u(0) = -1 < 0$:

- $0 < p < p_0$ two stationary points: $\theta_0 \in (0, \pi/2)$ minimum, $\theta_1 \in (\pi, 3/2\pi)$ maximum;

- $p = p_0$ three stationary points: $\theta_0 \in (0, \pi/2)$ minimum, $\theta_1 \in (\pi, 3/2\pi)$ maximum, $\theta_2 \in (3/2\pi, 2\pi)$ inflection point, such an inflection point will occur at the special angle value marked as $\phi_0$ in Figure 2;
- $p > p_0$ four stationary points: $\theta_0 \in (0, \pi/2)$ minimum, $\theta_1 \in (\pi, 3\pi/2)$ maximum, $\theta_2 \in (3\pi/2, 2\pi)$ minimum, $\theta_3 \in (3\pi/2, 2\pi)$ maximum.

# 4 The Dynamical Problem

Let us introduce the dynamical problem statement and its differential nonlinear equation.

## 4.1 The statement

In order to write the Lagrange equation, being the potential given by (3.2), it will be enough to compute the system kinetic energy $T(\theta)$. For this purpose, let us assume our system is but a planar rigid one, moving around to the $z$ fixed axis. Being:

$$J_z = J_{ring} + J_{rod} + J_{weight-block} = \frac{16}{3} mr^2,$$

then we have:

$$T(\theta) = \frac{8}{3} mr^2 \dot{\theta}^2, \tag{4.1}$$

after having inserted (3.2) and (4.1) in $L = T + U$ and then in Lagrange equation, we get:

$$\frac{16}{3} mr^2 \ddot{\theta} = -3mgr \sin\theta + mgr \cos\theta + \frac{9}{4} hr^2 \cos\theta \sin\theta$$

Such a nonlinear equation models our unforced, autonomous system and its integration will be our next step.

## 4.2 The Cauchy problem

First of all, we select some special value of the constants, what of course will not change its nature/behavior but will allow a better readability. We assume $r = 3g/16$ and put $a = 27h/64m$, so that the complete re-formulation of our problem becomes as follows.

$$\begin{cases} \ddot{\theta} = -3\sin\theta + \cos\theta + a\cos\theta\sin\theta \\ \theta(0) = 0 \\ \dot{\theta}(0) = 0 \end{cases} \tag{4.2}$$

with $a > 0$. Following the Weierstraß's method, see [19] pages 287-293 or [20] page 114, we consider:

$$\Phi(\theta) = 2\int_0^\theta (-3\sin\theta + \cos\theta + a\cos\theta\sin\theta)\mathrm{d}\theta = a\sin^2\theta + 2\sin\theta + 6\cos\theta - 6$$

and then

$$t = \int_0^\theta \frac{\mathrm{d}\theta}{\sqrt{a\sin^2\theta + 2\sin\theta + 6\cos\theta - 6}}. \tag{4.3}$$

When $a > 0$, we are going to detect the number of roots in $(0, 2\pi)$ of $\theta$ equation:

$$a\sin^2\theta + 2\sin\theta + 6\cos\theta - 6 = 0. \tag{4.4}$$

Notice that (4.4) has the solution $\theta = 0$: therefore the existence of another root implies the differential equation (4.2) has a periodic solution $\theta(t)$. Moreover, due to both zero initial conditions, the problem of the system movement, *for whichever value of the parameter a* , turns out to have nothing but periodical solutions: namely aperiodic motions will never happen.

In order to compute for variable $a$ the number of solutions of (4.4), let us count the cross number among the horizontal $a$-straight lines and the curve of equation

$$g(\theta) = \frac{6 - 6\cos\theta - 2\sin\theta}{\sin^2\theta}, \quad \theta \in (0, 2\pi)$$

which is represented below. After a computation we get:

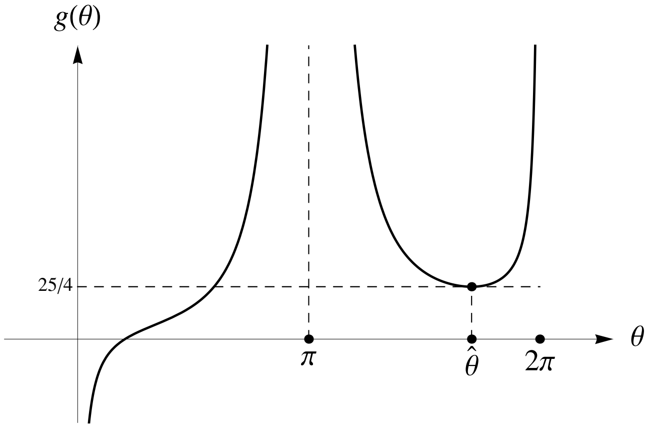


Figure 3: Plot of $g(\theta)$

$$g'(\theta) = \frac{2\left(3\sin^2\theta + 6\cos^2\theta - 6\cos\theta + \sin\theta\cos\theta\right)}{\sin^3\theta} = 0$$

$$\Longleftrightarrow \theta = \pi + 2\arctan 2 = \pi + \arccos\left(-\frac{3}{5}\right) := \hat{\theta}$$

$\hat{\theta}$ is the relative minimum corresponding to $25/4$. So that if $0 < a < 25/4$ equation (4.4) has only one solution in $(0, 2\pi)$ while for $a > 25/4$ equation (4.4) has three solutions in $(0, 2\pi)$. Notice that in the third sub-case too, illustrated with $a = 8$, the motion takes place from $\theta = 0$ till to the first intersection ($\theta < \pi$) between the curved line and the horizontal one; and back. Otherwise speaking: only the first two roots (of four), like in the subcritical case, will rule the oscillation amplitude even with $a$ beyond the critical value. The motion is then confined, or *trapped*, inside a classical potential well. The third and the fourth root are of course capable of affecting the supercritical case integration, and then the nature of solution which will come by equation (4.13) instead of (4.8) one. Finally, if $a = 25/4$ equation (4.4) will have two solutions.

Let us integrate the differential equation (4.2) in case $a = 4$, a case which is representative of all the solutions with $0 < a < 25/4$. Then a second root in $(0, 2\pi)$ of Weierstraß's function

$$4\sin^2\theta + 2\sin\theta + 6\cos\theta - 6$$

is $\theta = \frac{\pi}{2}$. Such an assumption does not reduce the generality to our motion analysis, but provides more quicker and readable computations. It was just for making our analysis tractable with $\theta(0) =$

$0$, that we give up to a deceptive generality, choosing some representative values of the process parameter $a$. We have:

$$t = \int_0^\theta \frac{\mathrm{d}\theta}{\sqrt{4\sin^2\theta + 2\sin\theta + 6\cos\theta - 6}}$$

In such a way our problem has been led to a quadrature. Changing the variable of integration, from $\theta$ to $\tau = \tan\frac{\theta}{2}$, we find:

$$t = \frac{1}{\sqrt{3}} \int_0^{\tan\frac{\theta}{2}} \frac{\mathrm{d}\tau}{\sqrt{\tau(1-\tau)\left[\left(\tau+\frac{1}{3}\right)^2 + \frac{2}{9}\right]}} \tag{4.5}$$

Now let us recall the definite integral in [21] section 3.147 formula 2 page 271:

$$\int_\beta^u \frac{\mathrm{d}x}{\sqrt{(\alpha - x)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F(\varphi, k), \tag{4.6}$$

where $\beta < u < \alpha$, and $F(\varphi, k)$ denotes the incomplete elliptic integral of first kind. Moreover $p^2 = (m-\alpha)^2 + n^2$, $q^2 = (m-\beta)^2 + m^2$ and:

$$\varphi(u) = 2\,\mathrm{arccot}\sqrt{\frac{q(\alpha - u)}{p(u-\beta)}}, \quad k = \frac{1}{2}\sqrt{\frac{(\alpha-\beta)^2 - (p-q)^2}{pq}}.$$

As a consequence, by identification:

$$\begin{array}{cccccc} \alpha & \beta & p & q & m & n \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 0 & \sqrt{2} & 1/\sqrt{3} & -1/3 & \sqrt{2}/3 \end{array}$$

we can explicitly write time equation (4.5) by means of an elliptic integral of first kind as:

$$t = \frac{1}{\sqrt[4]{6}} F\left(2\mathrm{arccot}\frac{\sqrt{\cot\frac{\theta}{2} - 1}}{\sqrt[4]{6}}, k_0\right), \quad k_0 = \sqrt{\frac{3-\sqrt{6}}{6}}. \tag{4.7}$$

By (4.7) we immediately get the motion period, doubling the value coming from $\theta = \frac{\pi}{2}$:

$$T_0 = 2\sqrt[4]{\frac{8}{3}}\,\boldsymbol{K}(k_0)$$

Equation (4.7) can be inverted through the Jacobi elliptic functions: in such a way the explicit solution of (4.2) for $a = 4$ is:

$$\theta(t) = 2\,\mathrm{arccot}\left(1 + \sqrt{6}\cot^2\left[\frac{1}{2}\mathrm{am}\left(\sqrt[4]{6}\,t, k_0\right)\right]\right) = 2\,\mathrm{arccot}\left(1 + \sqrt{6}\left(\frac{1 + \mathrm{cn}\left(\sqrt[4]{6}\,t, k_0\right)}{\mathrm{sn}\left(\sqrt[4]{6}\,t, k_0\right)}\right)^2\right), \tag{4.8}$$

being $\mathrm{am}(\cdot, k_0)$, $\mathrm{sn}(\cdot, k_0)$, $\mathrm{cn}(\cdot, k_0)$ some of the Jacobian elliptic functions of modulus $k_0$. The employ of the elliptic functions of Jacobi, very frequent in nonlinear mechanics, is not unique. As a recent example, the reader can see [22] where a partial KdV-like equation is solved through the Weierstraß elliptic function $\wp$.

Going to equation (4.2) for $a = 25/4$: let us change the variable as $\tan\frac{\vartheta}{2} = \tau$. Then an elementary integration leads to time equation:

$$t = 2\int_0^{\tan\frac{\theta}{2}} \frac{\mathrm{d}\tau}{(1+2\tau)\sqrt{\tau(4-3\tau)}} = \left[\frac{4}{\sqrt{11}}\arctan\sqrt{\frac{11\tau}{4-3\tau}}\right]_0^{\tan\frac{\theta}{2}} \tag{4.9}$$

By inversion we get another periodic, but not elliptic, angle output:

$$\theta(t) = 2\arctan\left(\frac{4\tan^2\left(\frac{\sqrt{11}}{4}\,t\right)}{11+3\tan^2\left(\frac{\sqrt{11}}{4}\,t\right)}\right), \tag{4.10}$$

whose period is:

$$T = 4\int_0^{\frac{4}{3}} \frac{\mathrm{d}\tau}{(1+2\tau)\sqrt{\tau(4-3\tau)}} = \frac{4\pi}{\sqrt{11}}$$

Finally, let us solve the motion whenever $a > \frac{25}{4}$. We select the special sub-case $a = 8$ finding the (4.3) roots without Cardano formulae. Through the usual change $\tan\frac{\vartheta}{2} = \tau$, time equation will take the form:

$$t = \frac{1}{\sqrt{3}}\int_0^{\tan\frac{\theta}{2}} \frac{\mathrm{d}\tau}{\sqrt{\tau(1+\tau)\left(\frac{2+\sqrt{7}}{3}-\tau\right)\left(\tau-\frac{2-\sqrt{7}}{3}\right)}} \tag{4.11}$$

Recalling the definite integral in [23] entry 256.00 page 120:

$$\int_b^y \frac{\mathrm{d}t}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}}\,F(\varphi,k), \tag{4.12}$$

where $a \geq y > b > c > d, andF(\varphi,k)$ denotes the incomplete elliptic integral of first kind with amplitude and squared modulus given by:

$$\varphi(y) = \arcsin\sqrt{\frac{(a-c)(y-b)}{(a-b)(y-c)}}, \quad k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}.$$

By identification

$$\begin{array}{ccccc} a & y & b & c & d \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \frac{2+\sqrt{7}}{3} & \tan\frac{\theta}{2} & 0 & \frac{2-\sqrt{7}}{3} & -1 \end{array}$$

we get:

$$t = \frac{\sqrt{2}}{\sqrt[4]{7}}F\left(\arcsin\left(\sqrt[4]{7}\sqrt{\frac{2\tan\frac{\theta}{2}}{1+(2+\sqrt{7})\tan\frac{\theta}{2}}}\right), k_1\right), \quad k_1 = \sqrt{\frac{7+\sqrt{7}}{14}}.$$

Then the motion period is given by:

$$T_1 = \frac{2\sqrt{2}}{\sqrt[4]{7}}\boldsymbol{K}(k_1).$$

By inversion, finally we find the explicit solution to (4.2) for $a = 8$

$$\theta(t) = 2\arctan\left(\frac{(\sqrt{7}-2)\,\mathrm{sn}\left(\frac{\sqrt[4]{7}}{\sqrt{2}}\,t, k_1\right)}{14-4\sqrt{7}-3\,\mathrm{sn}\left(\frac{\sqrt[4]{7}}{\sqrt{2}}\,t, k_1\right)}\right) \tag{4.13}$$

On the Figure 4, one reads the solutions to our main cases $a = 4,\ 25/4,\ 8$

# 5 Conclusions

The statical and dynamical analyses of an eccentric rotor led to a non-elementary goniometric equation and to a nonlinear differential equation.

The former is performed settling the equilibria and defining their nature, whose effective computation can be done numerically.

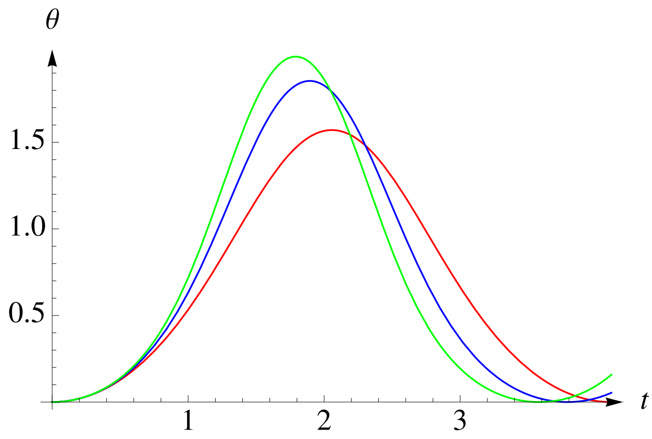


Figure 4: Solution curves to equation (4.2): $a = 4$ red, $a = 25/4$ blue, $a = 8$ green.

Coming to the latter: under both zero initial conditions, the system dynamics, for whichever $a$ value, *has nothing but periodical solutions*: each of them is ruled by the parameter $a = \frac{27h}{64m}$ relating the $h$ value of the spring rigidity to the inertia $m$. Three values of $a$ have been chosen as representative of its whole variation range. If $a$ increases along its range we will have progressively more damped and less massive systems. But the mass effect on $a$ is more than two times the damping: so that in such a sequence of non dissipative systems the fixed amount of energy induces maxima of angular displacement progressively higher, see Figure 4. We pass from the bell curve met as lowest, and having the lowest maximum, to other ones whose maxima are progressively higher. The dynamical analysis of all the possible cases has provided the angular displacement $\theta$ as jacobian elliptic function of time. The critical case, corresponding to $a = 25/4$, is ruled -only it- by a goniometric function. Closed form expressions of the period in each case have been obtained.

By a pure mechanical point of view, the reader is invited to see the Figure 1: all the motions will start from both zero angular position and speed: angle oscillations will arise, going on indefinitely. Notice that in the super-critical sub-case too, illustrated with $a = 8$, the motion is then trapped inside a classical potential well, as for the subcritical case. The different way, either (4.8) or (4.13), taken by the dynamical solution is ruled by the effective $a$ value solely.

In such a way the system turns out to behave like a generalized physical pendulum complicated by eccentricity and damping.

# Acknowledgment

The authors are indebted to their friend Aldo Scimone who drew the first figure of this paper: they hereby take opportunity to warmly thank him.

# Competing interests

The authors declare that no competing interests exist.

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