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Bayesian Analysis of Weibull-Lindley Distribution Using Different Loss Functions

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Authors' contributions

This work was carried out in collaboration among all authors. Authors IBE and OAB designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors BSY and KAM managed the analyses of the study. Author UAM managed the literature searches. All authors read and approved the final manuscript.

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ABSTRACT

In the present paper, a three-parameter Weibull-Lindley distribution is considered for Bayesian analysis. The estimation of a shape parameter of Weibull-Lindley distribution is obtained with the help of both the classical and Bayesian methods. Bayesian estimators are obtained by using Jeffrey's prior, uniform prior and Gamma prior under square error loss function, quadratic loss function and Precautionary loss function. Estimation by the method of Maximum likelihood is also discussed. These methods are compared by using mean square error through simulation study with varying parameter values and sample sizes.

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Keywords: Weibull-Lindley distribution; Bayesian method; priors; loss functions; MLE; simulation; MSE.

1. INTRODUCTION

There are several standard probability distributions that have been used over the years for modelling real-life datasets however research has shown that most of these distributions do not adequately model some of these heavily skewed datasets and therefore creating a problem in statistical theory and applications. Recently, numerous extended or compound probability distributions have proposed in the literature for modeling real-life situations and these compound distributions are found to be skewed, flexible and more better in statistical modeling compared to their standard counterparts [1-13].

Sequel to the fact above, [14] proposed and studied a new compound distribution known as "Weibull-Lindley distribution (WeiLinD)" with two shape parameters and a scale parameter. This distribution has been found to be skewed and flexible with different shapes and performed better than the Lomax-Lindley distribution [15], two-parameter Lindley distribution [16], transmuted Lindley distribution [17] and the conventional Lindley distribution [18] based on some applications of the models to four real-life datasets [14].

From [14], the probability density function (pdf), the cumulative distribution function (cdf), the survival function (sf), the hazard function (or failure rate) and quantile function (qf) of the Weibull-Lindley distribution are respectively defined as:

$$f(x)=\frac{\alpha\beta\theta^{2}(1+x)e^{-\theta x}\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\right\}^{\beta-1}e^{-\alpha\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\right\}^{\beta}}}{(\theta+1)\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]} \tag{1}$$

$$F(x;\theta,\alpha,\beta)=e^{-\alpha\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\right\}^{\beta}} \tag{2}$$

$$\mathrm{S}(\mathrm{x})=1-\mathrm{e}^{-\alpha\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]\mathrm{e}^{-\theta x}\right]\right\}^{\beta}} \tag{3}$$

$$h(x)=\frac{\alpha\beta\theta^{2}(1+x)e^{-\theta x}\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\right\}^{\beta-1}e^{-\alpha\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\right\}^{\beta}}}{(\theta+1)\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\left(1-e^{-\alpha\left\{-\log\left[1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right]\right\}^{\beta}}\right)} \tag{4}$$

and

$$Q(u)=-1-\frac{1}{\theta}-\frac{1}{\theta}W_{-1}\left(-(\theta+1)\left(1-\exp\left\{-\left(-\frac{\ln u}{\alpha}\right)^{\frac{1}{\beta}}\right\}\right)e^{-(\theta+1)}\right) \tag{5}$$

In which $x>0,\alpha,\beta,\theta>0$ and where α *and* β are the shape parameters and θ is the scale parameter of the Weibull-Lindley distribution.

A graphical representation of the probability density function, cumulative distribution function, survival function and hazard function of the Weibull-Lindley distribution for some selected parameter values presented in Fig. 1.

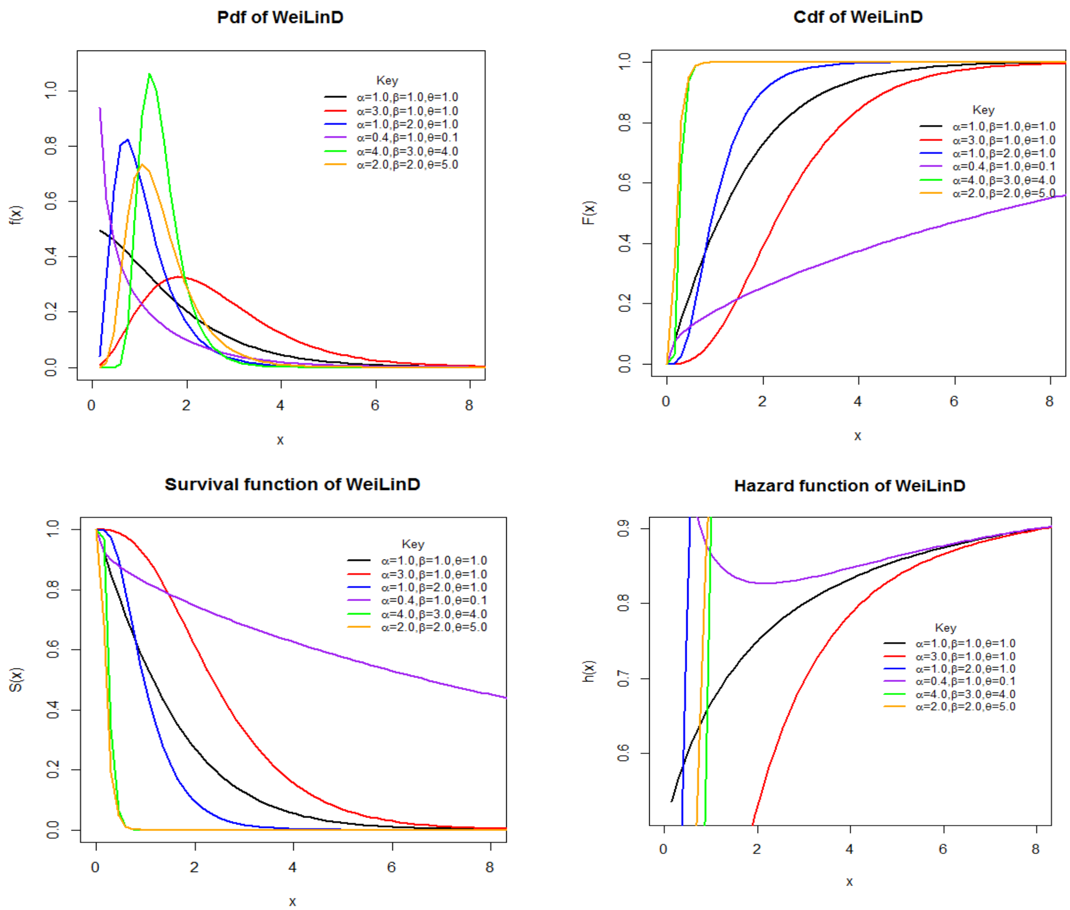

Fig. 1. Plots of the PDF, CDF, survival function and hazard function of the WeiLinD for selected parameter values

For details of the general behaviour of these functions, their properties, applications, different mathematical and statistical properties of the Weibull-Lindley distribution authors should check [14]. Due to the recorded performance of the WeiLinD in real life applications, it is deemed very important for the authors of this research to investigate and consider the most appropriate approach(s) for estimating the shape parameter of this distribution (WeiLinD) which will forever be useful during practical applications of this model.

There are two basic approaches to parameter estimation and these are the classical and the non-classical methods. The classical theory of estimation involves a situation where the parameters are considered to be constant but unknown whereas the parameters are considered to be unknown and random just like variables under non classical approach. The most widely used method in classical theory is the method of maximum likelihood estimation while the Bayesian estimation method is used in the non-classical theory. However, in most real life problems described by lifetime distributions, the parameters cannot be considered as constants in all the life testing period [19-21]. Following this narrative, it becomes obvious that the classical (frequentist) approach can no longer handle adequately problems of parameter estimation in life time models and therefore the need for non-classical or Bayesian estimation in life time models.

Estimation of parameters in a distribution differs by method from one parameter of the distribution to another and therefore this study aims at estimating one shape parameter of the Weibull-Lindley distribution using Bayesian approach and making a comparison between the Bayesian approach and the method of maximum likelihood estimation.

The aim of this article is to estimate a shape parameter of the WeiLinD using Bayesian approach assuming a uniform prior, Jeffrey's prior and gamma prior distributions with three loss functions. Next to this introductory section are the remaining contents of this article presented as follows: In Section 2, maximum likelihood estimator (MLE) for the shape parameter is obtained. In Section 3, Bayesian estimators based on different loss functions by assuming a uniform, Jeffrey's and gamma prior distributions are derived. The proposed estimators are compared in the relation of their mean squared error (MSE) in Section 4. Lastly, the conclusions is provided in Section 5.

2. MAXIMUM LIKELIHOOD ESTIMATION

Let $X_1, X_2, ..., X_n$ be a random sample from a population X of size 'n' independently and identically distributed random variables with probability density function $f(x),$. The likelihood is the joint probability function of the data, but viewed as a function of the parameters, treating the observed data as fixed quantities. Given that the values, $\underline{x} = (x_1, x_2, ..., x_n)$ are obtained independently from a WeiLinD with unknown parameters *α, β* and *Θ*, the likelihood function is given by:

$$L(\underline{x} \mid \alpha, \beta, \theta) = P(x_1, x_2, ..., x_n \mid \alpha, \beta, \theta) = \prod_{i=1}^{n} P(\underline{x} \mid \alpha, \beta, \theta) \tag{6}$$

The likelihood function, $L(\underline{x} \mid \alpha, \beta, \theta)$ based on the pdf of WeiLinD is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$ and it is given as:

$$L(\underline{X} \mid \alpha, \beta, \theta) = \frac{\left(\alpha\beta^{\alpha}\theta^{2}\right)^{n} \prod_{i=1}^{n}\left\{(1+x_i)e^{-\theta x_i}\right\} \prod_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta-1}}{(\theta+1)^{n} \prod_{i=1}^{n}\left\{1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right\} e^{\alpha \sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}}} \tag{7}$$

For the shape parameter of the WeiLinD α, the likelihood function is given by;

$$L(\underline{x} \mid \alpha) \propto \alpha^{n} \mathrm{e}^{-\alpha \sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}}$$

$$L(\underline{x} \mid \alpha) = K\alpha^{n} \mathrm{e}^{-\alpha \sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \tag{8}$$

Where $K = \dfrac{\prod_{i=1}^{n}\left\{(1+x_i)e^{-\theta x_i}\right\} \prod_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta-1}}{\beta^{-n\alpha}\theta^{-2n}(\theta+1)^{n} \prod_{i=1}^{n}\left\{1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right\}}$ is a constant which is independent of the shape parameter, α.

Let the log-likelihood function, $l = \log L(\underline{x} \mid \alpha)$, therefore

$$\log L(\underline{x} \mid \alpha) = n \log \alpha - \alpha \sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta} \tag{9}$$

Differentiating $\boldsymbol{l}$ partially with respect to *α, β* and *Θ* respectively gives;

$$\frac{\partial \log L\left(\underline{x} \mid \alpha\right)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta} = 0$$

And solving for $\hat{\alpha}$ gives;

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \tag{10}$$

where $\hat{\alpha}$ is the maximum likelihood estimator of the shape parameter, C. Details concerning the maximum likelihood estimators of the other two parameters of the WeiLinD can be found in [14].

3. BAYESIAN ESTIMATION

The Bayesian inference requires the appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one's subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s).

In this study, two non-informative priors (uniform and Jeffrey) and an informative prior (gamma) will be considered for estimating the shape parameter of the WeiLinD. These assumed prior distributions have been used widely by several authors including [22-29]. This study also considers three loss functions including square error, quadratic and precautionary loss functions which have also been used previously by some researchers such as [30-40] etc. The stated prior distributions and loss functions are defined as follows:

a. **The uniform prior is defined as:**

$$p(\alpha) \propto 1; 0 < \alpha < \infty \tag{11}$$

b. **Also, the Jeffrey's prior is defined as:**

$$p(\alpha) \propto \frac{1}{\alpha}; 0 < \alpha < \infty \tag{12}$$

c. **Also, the gamma prior is defined as:**

$$P(\alpha) = \frac{a^b}{\Gamma(b)}\alpha^{b-1}e^{-a\alpha} \tag{13}$$

i. Squared Error Loss Function (SELF)

The squared error loss function relating to the shape parameter α is defined as:

$$L(\alpha, \alpha_{SELF}) = (\alpha - \alpha_{SELF})^2 \tag{14}$$

where α_{SELF} is the estimator of the parameter α under *SELF*.

ii. Quadratic Loss Function (QLF)

The quadratic loss function is defined from [41] as

$$L\left(\alpha,\alpha_{QLF}\right)=\left(\frac{\alpha-\alpha_{QLF}}{\alpha}\right)^2 \quad (15)$$

where α_{QLF} is the estimator of the parameter α under *QLF*.

iii. Precautionary Loss Function (PLF)

The precautionary loss function (PLF) introduced by [42] is an asymmetric loss function and is defined as

$$L\left(\alpha_{PLF},\alpha\right)=\frac{\left(\alpha_{PLF}-\alpha\right)^2}{\alpha_{PLF}} \quad (16)$$

where α_{PLF} is the estimator of the shape parameter α under *PLF*.

The posterior distribution of a parameter is the distribution of the parameter after observing the available data and it is obtained by using Bayes' theorem in relation to the shape parameter α, likelihood function and prior distribution as follows:

$$P\left(\alpha\,|\,\underline{x}\right)=\frac{P\left(\alpha,\underline{x}\right)}{P\left(\underline{x}\right)}=\frac{P\left(\underline{x}\,|\,\alpha\right)P\left(\alpha\right)}{P\left(\underline{x}\right)}=\frac{P\left(\underline{x}\,|\,\alpha\right)P\left(\alpha\right)}{\int P\left(\underline{x}\,|\,\alpha\right)P\left(\alpha\right)d\alpha}=\frac{L\left(\underline{x}\,|\,\alpha\right)P\left(\alpha\right)}{\int L\left(\underline{x}\,|\,\alpha\right)P\left(\alpha\right)d\alpha} \quad (17)$$

where $P\left(\underline{x}\right)$ is the marginal distribution of X and $P\left(\underline{x}\right)=\sum_{x}^{\infty}p(\alpha)L(\underline{x}\,|\,\alpha)$ when the prior distribution of α is discrete and $P\left(\underline{x}\right)=\int_{-\infty}^{\infty}p(\alpha)L(\underline{x}\,|\,\alpha)d\alpha$ when the prior distribution of α is continuous? Also note that $p\left(\alpha\right)$ and $L\left(\underline{x}\,|\,\alpha\right)$ are the prior distribution and the Likelihood function respectively.

3.1 Bayesian Analysis under Uniform Prior with Three Loss Functions

The posterior distribution of the shape parameter α assuming a uniform prior distribution is obtained from equation (17) using integration by substitution method as:

$$P\left(\alpha\,|\,\underline{x}\right)=\frac{\left(\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}\right)^{n+1}}{\Gamma\left(n+1\right)\alpha^{-n}\mathrm{e}^{\alpha\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}}} \quad (18)$$

Now the Bayes estimators under uniform prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF}=E\left(\alpha\right)=E\left(\alpha\,|\,\underline{x}\right)=\int_{0}^{\infty}\alpha P\left(\alpha\,|\,\underline{x}\right)d\alpha=\frac{n+1}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \quad (19)$$

$$\alpha_{QLF}=\frac{E\left(\alpha^{-1}\,|\,\underline{x}\right)}{E\left(\alpha^{-2}\,|\,\underline{x}\right)}=\frac{\int_{0}^{\infty}\alpha^{-1}P\left(\alpha\,|\,\underline{x}\right)d\alpha}{\int_{0}^{\infty}\alpha^{-2}P\left(\alpha\,|\,\underline{x}\right)d\alpha}=\frac{\left(n-1\right)}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \quad (20)$$

and

$$\alpha_{PLF} = \left\{E\left(\alpha^2 \mid \underline{x}\right)\right\}^{\frac{1}{2}} = \left\{\int_0^\infty \alpha^2 P\left(\alpha \mid \underline{x}\right) d\alpha\right\}^{\frac{1}{2}} = \frac{\left[(n+1)(n+2)\right]^{0.5}}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \quad (21)$$

3.2 Bayesian Analysis under Jeffrey's Prior with Three Loss Functions

The posterior distribution of the shape parameter α for a given data assuming a Jeffrey's prior distribution is obtained from equation (17) using integration by substitution method as:

$$P\left(\alpha \mid \underline{x}\right) = \frac{\left(\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}\right)^{n}}{\Gamma(n)\alpha^{-n-1}\mathrm{e}^{\alpha\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}}} \quad (22)$$

Again the Bayes estimators under Jeffrey's prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF} = E(\alpha) = E\left(\alpha \mid \underline{x}\right) = \int_0^\infty \alpha P\left(\alpha \mid \underline{x}\right) d\alpha = \frac{n}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \quad (23)$$

$$\alpha_{QLF} = \frac{E\left(\alpha^{-1} \mid \underline{x}\right)}{E\left(\alpha^{-2} \mid \underline{x}\right)} = \frac{\int_0^\infty \alpha^{-1} P\left(\alpha \mid \underline{x}\right) d\alpha}{\int_0^\infty \alpha^{-2} P\left(\alpha \mid \underline{x}\right) d\alpha} = \frac{(n-2)}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \quad (24)$$

and

$$\alpha_{PLF} = \left\{E\left(\alpha^2 \mid \underline{x}\right)\right\}^{\frac{1}{2}} = \left\{\int_0^\infty \alpha^2 P\left(\alpha \mid \underline{x}\right) d\alpha\right\}^{\frac{1}{2}} = \frac{\left[n(n+1)\right]^{0.5}}{\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \quad (25)$$

3.3 Bayesian Analysis under Gamma Prior with Three Loss Functions

The posterior distribution of the shape parameter α for a given data assuming a gamma prior distribution is obtained from equation (17) using integration by substitution method as

$$P\left(\alpha \mid \underline{x}\right) = \frac{\left(a+\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}\right)^{n+b}}{\Gamma(n+b)\alpha^{-(n+b-1)}e^{\alpha\left(a+\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}\right)}} \quad (26)$$

Table 1. Estimates and mean squared errors (within parenthesis) for $\hat{\alpha}$ under uniform prior

Sample size (n)	Parameter (True value)					Methods of estimation			
	α	β	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
25	0.5	0.5	0.5	1.0	2.0	0.6728 (1.1980)	0.6997 (1.3033)	0.6459 (1.0978)	0.7130 (1.3574)
	0.5	1.0	0.5	1.0	1.0	0.5202 (0.0123)	0.5410 (0.0146)	0.4994 (0.0110)	0.5513 (0.0160)
	1.0	1.0	1.0	1.0	1.0	1.0457 (0.0505)	1.0875 (0.0600)	1.0039 (0.0446)	1.1082 (0.0661)
	2.0	1.0	1.0	1.0	1.0	2.0914 (0.2021)	2.1750 (0.2401)	2.0077 (0.1786)	2.2165 (0.2644)
	1.0	1.0	2.0	1.5	1.0	1.0457 (0.0505)	1.0875 (0.0600)	1.0039 (0.0446)	1.1082 (0.0661)
50	0.5	0.5	0.5	1.0	2.0	0.2594 (0.8420)	0.2646 (0.8712)	0.2542 (0.8135)	0.2672 (0.8860)
	0.5	1.0	0.5	1.0	1.0	0.5095 (0.0055)	0.5197 (0.0060)	0.4993 (0.0052)	0.5248 (0.0063)
	1.0	1.0	1.0	1.0	1.0	1.0235 (0.0220)	1.0440 (0.0242)	1.0030 (0.0206)	1.0542 (0.0257)
	2.0	1.0	1.0	1.0	1.0	2.047 (0.088)	2.088 (0.097)	2.0061 (0.0824)	2.1083 (0.1027)
	1.0	1.0	2.0	1.5	1.0	1.0235 (0.0220)	1.0440 (0.0242)	1.0030 (0.0206)	1.0542 (0.0257)
100	0.5	0.5	0.5	1.0	2.0	0.0263 (0.3439)	0.0266 (0.3460)	0.0261 (0.3417)	0.0267 (0.3471)
	0.5	1.0	0.5	1.0	1.0	0.5046 (0.0026)	0.5096 (0.0028)	0.4996 (0.0026)	0.5122 (0.0029)
	1.0	1.0	1.0	1.0	1.0	1.0079 (0.0102)	1.0180 (0.0106)	0.9979 (0.0099)	1.0231 (0.0109)
	2.0	1.0	1.0	1.0	1.0	2.0159 (0.0407)	2.0361 (0.0425)	1.9957 (0.0396)	2.0461 (0.0438)
	1.0	1.0	2.0	1.5	1.0	1.0079 (0.0102)	1.0180 (0.0106)	0.9979 (0.0099)	1.0231 (0.0109)
150	0.5	0.5	0.5	1.0	2.0	0.0083 (0.2882)	0.0084 (0.2888)	0.0083 (0.2876)	0.0084 (0.2890)
	0.5	1.0	0.5	1.0	1.0	0.5030 (0.0017)	0.5064 (0.0018)	0.4996 (0.0017)	0.5080 (0.0018)
	1.0	1.0	1.0	1.0	1.0	1.0066 (0.0068)	1.0133 (0.0071)	0.9999 (0.0067)	1.0166 (0.0072)
	2.0	1.0	1.0	1.0	1.0	2.0132 (0.0273)	2.0266 (0.0282)	1.9997 (0.0268)	2.0333 (0.0288)
	1.0	1.0	2.0	1.5	1.0	1.0066 (6.8e-3)	1.0133 (7.1e-3)	0.9999 (6.7e-3)	1.0166 (7.2e-3)
200	0.5	0.5	0.5	1.0	2.0	0.00 (0.25)	0.00 (0.25)	0.00 (0.25)	0.00 (0.25)
	0.5	1.0	0.5	1.0	1.0	0.5028 (1.3e-3)	0.5054 (1.3e-3)	0.5003 (1.3e-3)	0.5066 (1.3e-3)
	1.0	1.0	1.0	1.0	1.0	1.0057 (5.3e-3)	1.0107 (5.4e-3)	1.0006 (5.2e-3)	1.0132 (5.5e-3)
	2.0	1.0	1.0	1.0	1.0	2.011 (0.0211)	2.0214 (0.0217)	2.0013 (0.0208)	2.0264 (0.0220)
	1.0	1.0	2.0	1.5	1.0	1.0057 (5.3e-3)	1.0107 (5.4e-3)	1.0006 (5.2e-3)	1.0132 (5.5e-3)

MLE = Maximum likelihood estimator, SELF = Square error loss function, QLF = Quadratic loss function, PLF = Precautionary loss function

Table 2. Estimates and mean squared errors (within parenthesis) for $\hat{\alpha}$ under Jeffrey's prior

Sample size (n)	Parameter (True value)					Methods of estimation			
	α	β	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
25	0.5	0.5	0.5	1.0	2.0	0.6728 (1.1980)	0.6728 (1.1980)	0.6190 (1.0028)	0.6861 (1.2495)
	0.5	1.0	0.5	1.0	1.0	0.5202 (0.0123)	0.5202 (0.0123)	0.4786 (0.0105)	0.5305 (0.0133)
	1.0	1.0	1.0	1.0	1.0	1.0457 (0.0505)	1.0457 (0.0505)	0.9620 (0.0424)	1.0664 (0.0548)
	2.0	1.0	1.0	1.0	1.0	2.0914 (0.2021)	2.0914 (0.2021)	1.9241 (0.1697)	2.1328 (0.2191)
	1.0	1.0	2.0	1.5	1.0	1.0457 (0.0505)	1.0457 (0.0505)	0.9620 (0.0424)	1.0664 (0.0548)
50	0.5	0.5	0.5	1.0	2.0	0.2594 (0.8420)	0.2594 (0.8420)	0.2491 (0.7856)	0.2620 (0.8565)
	0.5	1.0	0.5	1.0	1.0	0.5095 (0.0055)	0.5095 (0.0055)	0.4891 (0.0051)	0.5146 (0.0057)
	1.0	1.0	1.0	1.0	1.0	1.0235 (0.0220)	1.0235 (0.0220)	0.9826 (0.0201)	1.0337 (0.0230)
	2.0	1.0	1.0	1.0	1.0	2.047 (0.088)	2.047 (0.088)	1.9651 (0.0803)	2.0674 (0.0920)
	1.0	1.0	2.0	1.5	1.0	1.0235 (0.0220)	1.0235 (0.0220)	0.9826 (0.0201)	1.0337 (0.0230)
100	0.5	0.5	0.5	1.0	2.0	0.0263 (0.3439)	0.0263 (0.3439)	0.0258 (0.3396)	0.0265 (0.3449)
	0.5	1.0	0.5	1.0	1.0	0.5046 (0.0026)	0.5046 (0.0026)	0.4945 (0.0026)	0.5071 (0.0027)
	1.0	1.0	1.0	1.0	1.0	1.0079 (0.0102)	1.0079 (0.0102)	0.9878 (0.0099)	1.0130 (0.0104)
	2.0	1.0	1.0	1.0	1.0	2.0159 (0.0407)	2.0159 (0.0407)	1.9756 (0.0394)	2.0259 (0.0415)
	1.0	1.0	2.0	1.5	1.0	1.0079 (0.0102)	1.0079 (0.0102)	0.9878 (0.0099)	1.0130 (0.0104)
150	0.5	0.5	0.5	1.0	2.0	0.0083 (0.2882)	0.0083 (0.2882)	0.0082 (0.2871)	0.0084 (0.2885)
	0.5	1.0	0.5	1.0	1.0	0.5030 (0.0017)	0.5030 (0.0017)	0.4963 (0.0017)	0.5047 (0.0017)
	1.0	1.0	1.0	1.0	1.0	1.0066 (0.0068)	1.0066 (0.0068)	0.9932 (0.0067)	1.0099 (0.0069)
	2.0	1.0	1.0	1.0	1.0	2.0132 (0.0273)	2.0132 (0.0273)	1.9863 (0.0266)	2.0199 (0.0277)
	1.0	1.0	2.0	1.5	1.0	1.0066 (6.8e-3)	1.0066 (6.8e-3)	0.9932 (6.7e-3)	1.0099 (6.9e-3)
200	0.5	0.5	0.5	1.0	2.0	0.00 (0.25)	0.00 (0.25)	0.00 (0.25)	0.00 (0.25)
	0.5	1.0	0.5	1.0	1.0	0.5028 (1.3e-3)	0.5028 (1.3e-3)	0.4978 (1.2e-3)	0.5041 (1.3e-3)
	1.0	1.0	1.0	1.0	1.0	1.0057 (5.3e-3)	1.0057 (5.3e-3)	0.9956 (5.2e-3)	1.0082 (5.3e-3)
	2.0	1.0	1.0	1.0	1.0	2.0114 (0.0211)	2.0114 (0.0211)	1.9912 (0.0206)	2.0164 (0.0214)
	1.0	1.0	2.0	1.5	1.0	1.0057 (5.3e-3)	1.0057 (5.3e-3)	0.9956 (5.2e-3)	1.0082 (5.3e-3)

MLE = Maximum likelihood estimator, SELF = Square error loss function, QLF = Quadratic loss function, PLF = Precautionary loss function

Table 3. Estimates and mean squared errors (within parenthesis) for $\hat{\alpha}$ under gamma prior

Sample size (n)	Parameter (True value)					Methods of estimation			
	α	β	θ	a	b	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{SELF}$	$\hat{\alpha}_{QLF}$	$\hat{\alpha}_{PLF}$
25	0.5	0.5	0.5	1.0	2.0	0.6728 (1.1980)	0.6629 (1.1529)	0.6138 (0.9787)	0.6751 (1.1988)
	0.5	1.0	0.5	1.0	1.0	0.5202 (0.0123)	0.5295 (0.0126)	0.4888 (0.0102)	0.5396 (0.0138)
	1.0	1.0	1.0	1.0	1.0	1.0457 (0.0505)	1.0421 (0.0456)	0.9619 (0.0388)	1.0619 (0.0494)
	2.0	1.0	1.0	1.0	1.0	2.0914 (0.2021)	2.0009 (0.1481)	1.8470 (0.1496)	2.0390 (0.1553)
	1.0	1.0	2.0	1.5	1.0	1.0457 (0.0505)	1.0208 (0.0407)	0.9423 (0.0376)	1.0403 (0.0434)
50	0.5	0.5	0.5	1.0	2.0	0.2594 (0.8420)	0.2532 (0.8068)	0.2435 (0.7554)	0.2556 (0.8199)
	0.5	1.0	0.5	1.0	1.0	0.5095 (0.0055)	0.5144 (0.0056)	0.4942 (0.0050)	0.5194 (0.0059)
	1.0	1.0	1.0	1.0	1.0	1.0235 (0.0220)	1.0226 (0.0210)	0.9825 (0.0192)	1.0326 (0.0220)
	2.0	1.0	1.0	1.0	1.0	2.047 (0.088)	2.0043 (0.0756)	1.9257 (0.0752)	2.0239 (0.0776)
	1.0	1.0	2.0	1.5	1.0	1.0235 (0.0220)	1.0123 (0.0198)	0.9726 (0.0189)	1.0222 (0.0205)
100	0.5	0.5	0.5	1.0	2.0	0.0263 (0.3439)	0.0257 (0.3386)	0.0252 (0.3347)	0.0258 (0.3396)
	0.5	1.0	0.5	1.0	1.0	0.5046 (0.0026)	0.5071 (0.0027)	0.4970 (0.0025)	0.5096 (0.0027)
	1.0	1.0	1.0	1.0	1.0	1.0079 (0.0102)	1.0078 (0.0100)	0.9878 (0.0097)	1.0127 (0.0102)
	2.0	1.0	1.0	1.0	1.0	2.0159 (0.0407)	1.9954 (0.0380)	1.9559 (0.0385)	2.0053 (0.0384)
	1.0	1.0	2.0	1.5	1.0	1.0079 (0.0102)	1.0027 (0.0097)	0.9829 (0.0096)	1.0077 (0.0099)
150	0.5	0.5	0.5	1.0	2.0	0.0083 (0.2882)	0.0081 (0.2863)	0.0080 (0.2852)	0.0082 (0.2865)
	0.5	1.0	0.5	1.0	1.0	0.5030 (0.0017)	0.5046 (0.0017)	0.4980 (0.0017)	0.5063 (0.0018)
	1.0	1.0	1.0	1.0	1.0	1.0066 (0.0068)	1.0065 (0.0067)	0.9932 (0.0066)	1.0098 (0.0068)
	2.0	1.0	1.0	1.0	1.0	2.0132 (0.0273)	1.9996 (0.0261)	1.9731 (0.0261)	2.0062 (0.0263)
	1.0	1.0	2.0	1.5	1.0	1.0066 (0.0068)	1.0031 (0.0066)	0.9898 (0.0065)	1.0064 (0.0067)
200	0.5	0.5	0.5	1.0	2.0	0.00 (0.25)	0.00 (0.2500)	0.00 (0.2500)	0.00 (0.2500)
	0.5	1.0	0.5	1.0	1.0	0.5028 (1.3e-3)	0.5041 (1.3e-3)	0.4991 (1.2e-3)	0.5053 (1.3e-3)
	1.0	1.0	1.0	1.0	1.0	1.0057 (5.3e-3)	1.0056 (5.2e-3)	0.9956 (5.1e-3)	1.0081 (5.3e-3)
	2.0	1.0	1.0	1.0	1.0	2.0114 (0.0211)	2.0012 (0.0204)	1.9813 (0.0203)	2.0062 (0.0205)
	1.0	1.0	2.0	1.5	1.0	1.0057 (5.3e-3)	1.0031 (5.2e-3)	0.9931 (5.1e-3)	1.0056 (5.2e-3)

MLE = Maximum likelihood estimator, SELF = Square error loss function, QLF = Quadratic loss function, PLF = Precautionary loss function

Also the Bayes estimators under gamma prior using *SELF*, *QLF* and *PLF* are given respectively as:

$$\alpha_{SELF} = \frac{n+b}{a+\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \tag{27}$$

$$\alpha_{QLF} = \frac{n+b-2}{a+\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \tag{28}$$

and

$$\alpha_{PLF} = \frac{\left[(n+b+1)(n+b)\right]^{0.5}}{a+\sum_{i=1}^{n}\left\{-\log\left[1-\left[1+\frac{\theta x_i}{\theta+1}\right]e^{-\theta x_i}\right]\right\}^{\beta}} \tag{29}$$

4. RESULTS AND DISCUSSION

In this section, Monte Carlo simulation with R software under 10,000 replications is considered to generate random samples of sizes n = (25, 50, 100, 150, 200) from Weibull-Lindley distribution using the quantile function (inverse transformation method of simulation) under the following combination of parameter values: $\alpha = 0.5, \beta = 0.5, \theta = 0.5, a = 1$ and $b = 2$; $\alpha = 0.5, \beta = 1.0, \theta = 0.5, a = 1.0$ and $b = 1.0$; $\alpha = 1.0, \beta = 1.0, \theta = 1.0, a = 1.0$ and $b = 1.0$, $\alpha = 2.0, \beta = 1.0, \theta = 1.0, a = 1.0$ and $b = 1.0$ and $\alpha = 1.0, \beta = 1.0, \theta = 2.0, a = 1.5$ and $b = 1.0$. The following tables present the results of our simulation study by listing the average estimates of the shape parameter with their respective Mean Square Errors (MSEs) under the appropriate estimation methods which include the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function (*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under Uniform Jeffrey and gamma priors respectively. The criterion for evaluating the performance of the estimators in this study is the Mean Square Error (MSE): $MSE = \frac{1}{n}E\left(\hat{\alpha}-\alpha\right)^2$.

Considering the results from Tables 1-3, it is revealed that the estimators of the shape parameter using QLF under Gamma, uniform and Jeffrey priors is better than the other estimators based on the fact that it has a smaller MSE despite the changes in the samples and chosen parameter values. This consistent smaller value of the MSE for Bayesian estimators (using QLF under Uniform, Jeffrey and gamma priors) is an indication that the method is the more efficient for estimating this shape parameter compared to the Method of Maximum Likelihood estimation (*MLE*) and Bayesian with the other two loss functions. Also, making comparison across the prior distributions it is revealed that the QLF under the gamma prior has the smallest MSEs compared to the QLF under uniform and Jeffrey priors irrespective of the allocated parameter values and the sample sizes and this performance of the QLF is found to be consistent despite all odds.

On a general note, the results in Tables 1-3 has shown that the average estimates of the shape parameter tend to be closer to the true parameter value when sample size increases and the mean square errors (MSEs) all decrease as sample size increases which satisfies the first-order asymptotic theory. Also, Bayesian estimators and maximum likelihood estimators (MLEs) all become better when the sample size increases. In fact, for very large sample sizes the performances of these estimators are observed to be the same for both methods of estimation.

5. CONCLUSION

This study has derived Bayesian estimators for a shape parameter of Weibull-Lindley distribution by assuming Uniform, Jeffrey and gamma prior distributions with three loss functions which include Squared Error Loss Function, Quadratic Loss Function and Precautionary Loss Function.

The Posterior distributions and Bayes estimators of this parameter are derived and simplified using the priors and loss functions respectively. The performance of these estimators have been assess on the basis of their mean square errors using the inverse transformation method of Monte Carlo Simulations for different parameter values and sample sizes. The results of the simulation and comparison show that using the *QLF* gives estimators with the smallest MSEs under all the prior distributions (gamma, Jeffreys and uniform). Most importantly, it is revealed that Bayesian Method using Quadratic Loss Function under gamma prior produces the best estimators of the shape parameter compared to estimators using Maximum Likelihood method, Squared Error Loss Function and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors irrespective of the selected values of the parameters and the allocated sample sizes. This research also found that the variation in the values of the other two parameters of the distribution do not influence the performance of the estimators of the estimated shape parameter, however, it is suggested that since this study considers only a shape parameter of the WeiLinD, upcoming research works should look at the remaining two parameters of the distribution because in practical applications of this model it will be very necessary to identify and understand the best method for estimating all the unknown parameters of the model.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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