Mean Square Asymptotic Boundedness of Stochastic Complex Networks via Impulsive Control

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

Abstract

In this paper, the mean square asymptotic boundedness of a class of stochastic complex systems with different dynamic nodes represented by Ito stochastic differential equations is studied. By using the Lyapunov function and Ito formula, the mean square asymptotic boundedness and mean square asymptotic stability conditions of stochastic complex systems with different dynamic nodes are obtained.

Keywords: Stochastic complex networks; mean square asymptotic boundedness; Ito formula.

1 Introduction

Complex systems are composed of the complex links between the nodes and the nodes of a kind of important dynamic system. Watts and Strogatz' work in the famous small world network model was established in 1998 and described its features. And complex systems in the real world are widespread, such as the food web, communication network, mobile network, the world wide web, metabolic system, transmission system, etc. At the same time, stochastic phenomenon is a common natural phenomena in the nature, the influence of a lot of the real systems can't avoid it, such as complex network signals are often in the process of transfer under the influence of stochastic factors in the surrounding environment and become a stochastic process by noise [1]. And because of the existence of random interference factors, such as space systems [7][8] nuclear

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reactor[9], chemical reaction system, neural network[10], industrial control system[11], financial and economic system[12][13] and so on, many real systems are impossible to use ordinary differential equation to accurately depict.

Stability is an important part of the dynamic characteristics in the study of stochastic complex networks, is also one of the main goals of engineering design. Because the concept of stability and the classification of stochastic system are different, therefore there are many different methods to study the stability of stochastic complex networks. At present, many results on the basic theory and stability theory of stochastic complex networks have been reported. At the same time, boundedness is also very important in the study of stochastic complex networks. But relatively few work was done on the boundedness of complex networks. So this article will discuss the mean square asymptotic boundedness of stochastic complex networks.

Inspired by the literature [14], this paper will extend the results in [14] to the stochastic case. By using the Lyapunov function and Ito formula, the mean square asymptotic boundedness and mean square asymptotic stability conditions of stochastic complex systems with different dynamic nodes are obtained. Results indicate that when the stochastic complex network is unbounded, impulsive control can be used to make it bounded.

2 Model Description and Preliminaries

Let $\mathbf{R}^n$ define the n-dimensional Euclidean space, E_n define the n-dimensional unit matrix, $\otimes$ define the Kronecker product of two matrices, $\mathbf{N} = \{1, 2, 3, \cdots\}$ and $\mathbf{R}_+ = [0, \infty)$. Let $\|\cdot\|$ define the Euclidean norm, and $\mathbf{N}_{[a,b]} = \{a, a+1, \cdots, b\}$ where $a < b$, and a and b are integral numbers. If A is a vector or matrix, its transpose is defined by A^T. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ define the minimum and the maximum eigenvalues of the corresponding matrix, respectively. And $trace(\cdot)$ defines the trace of the matrix.

Consider the stochastic complex network of N nodes, and each node has an n-dimensional dynamical system i.e.

$$dx_i(t) = \left[f\left(t, x_i(t)\right) + \sum_{j=1}^N C_{ij}(t) A x_j(t) \right] dt + g\left(t, x_i(t)\right) dw(t), i = 1, 2, \cdots, \mathbf{N}. \tag{1}$$

where $x_i(t) = \left(x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)\right)^T \in \mathbf{R}^n$ represents the state vector of the *ith* node of the network at time t, $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous nonlinear vector valued function, $g\left(x_i(t)\right) : \mathbf{R}^n \to \mathbf{R}^n$ is the noise density function, $w(t)$ is the n-dimensional Brownian motion, and $dw_i(t)$ is independent of $dw_j(t)$, $C(t) = \left(c_{ij}(t)\right)_{N \times N}$ is the coupling configuration matrix of the networks, and $A \in \mathbf{R}^{n \times n}$ is the inner connecting matrix in each node. For simplicity of further discussion, we separate the linear part from the nonlinear part of f as

$$f\left(t, x_i(t)\right) = F(t) x_i + \phi(t, x_i), i = 1, 2, \cdots, \mathbf{N}, \tag{2}$$

where $F(t)$ is the corresponding time-variable matrices. Using the Kronecker product, network (1) can be rewritten as

$$dX(t) = \left[D(t)X(t) + \Phi(t,X) + (C(t) \otimes A)X(t) \right]dt + g(t,X)dw(t),$$

(3)

where $X(t) = (x_1(t), x_2(t), \cdots, x_N(t))^T$, $D(t) = diag\left\{\underbrace{F(t), F(t), \cdots, F(t)}_{N}\right\}$, and

$\Phi(t,X) = (\phi(t, \quad x_1), \phi(t,x_2)\ldots,\phi(t,x_N))^T$.

Let the plant ρ describe the evolution process $(t, X(t))$, where $X(t) \in \mathrm{R}^{nN}$ is the state variable of **Error! Reference source not found.**

2.1 Definition

If there exists a positive constant $\beta > 0$, such that $\limsup_{t \to \infty} E\|x(t,t_0,\varphi)\|^2 \leq \beta$, then the solution of the system **Error! Reference source not found.** is said to be mean square asymptotical bounded.

2.2 Definition [15]

Sequence $\{t_k, U(k, X(t_k))\}$ is said to be a law of ρ if $X(t_k^+) = X(t_k^-) + U(k, X(t_k^-))$, $k \in \mathrm{N}, t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots, \lim_{k \to \infty} t_k = \infty$.

2.3 Definition

Let $A = (a_{ij})_{p \times q} \in \mathrm{R}^{p \times q}$ and $B = (b_{ij})_{m \times n} \in \mathrm{R}^{m \times n}$, then the following block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathrm{R}^{pm \times qn},$$

is the Kronecker product of A and B .

At the same time, for any matrix and constant, Kronecker product has the following properties:

$(1)(A \otimes B)^T = A^T \otimes B^T$;

$(2)(aA) \otimes B = A \otimes (aB)$;

$(3)(A+B) \otimes C = A \otimes C + B \otimes C$;

$(4)(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Our main aim is to design a law such that system **Error! Reference source not found.** under the law is mean square asymptotical bounded. We design the following law:

$$U\left(k, X\left(t_k^-\right)\right) = B_k X\left(t_k^-\right), k \in \mathrm{N}, \tag{2}$$

where the matrices $B_k = diag\{B_{1k}, \cdots, B_{Nk}\} \in \mathrm{R}^{nN \times nN}\left(B_{ik} \in \mathrm{R}^{n \times n}, i \in \mathrm{N}_{[1,N]}, k \in \mathrm{N}\right)$ are the feedback gain at moment t_k. Then, we can obtain the impulsive stochastic complex networks as follows:

$$\begin{cases} dX(t) = \left[D(t)X(t) + \Phi(t, X) + \left(C(t) \otimes A\right)X(t)\right]dt \\ \quad + g(t, X)dw(t), t \neq t_k, t \geq t_0 \\ \qquad \Delta X\left(t_k^+\right) = B_k X\left(t_k\right), k \in \mathrm{N} \\ \qquad\quad X\left(t_0^+\right) = X_0 \end{cases}, \tag{3}$$

where $\Delta X\left(t_k^+\right) = X\left(t_k^+\right) - X\left(t_k^-\right)$, $X\left(t_k^+\right) = \lim_{t \to t_k^+} X(t)$, and any solution of (3) is left continuous at each t_k, i.e. $X\left(t_k^-\right) = X\left(t_k\right)$. Furthermore, we suppose that system (3) satisfies the following hypotheses.

2.4 Assumption

There exist constants $L > 0$, $\hat{L} > 0$ and $J > 0, \hat{J} > 0$, and a locally integrable function $h(t) > 0$ and positive definite matrices P_1, such that

$$\phi(t, x_i)^T P x_i \leq h(t)\left[L x_i^T P x_i + J\right], i \in \mathrm{N}_{[1,N]}, \tag{4}$$

$$trace\left[g^T(t, x_i) P_1 g(t, x_i)\right] \leq h(t)\left[\hat{L} x_i^T P_1 x_i + \hat{J}\right], i \in \mathrm{N}_{[1,N]}. \tag{5}$$

2.5 Remark

From Assumption 2.1, we have

$$\Phi(t, X)^T PX \leq h(t)\left(L X^T PX + J\right),$$

$$trace\left[g^T(t, X) Pg(t, X)\right] \leq h(t)\left[\hat{L} X^T PX + \hat{J}\right],$$

where $P = diag\left\{\underbrace{P, P, \cdots, P}_{N}\right\}$.

2.6 Assumption

There exists a constant $\alpha \in \mathbb{R}$ such that

$$\lambda_{\max}\left(P^{-1}\left(D(t)^T P + PD(t)\right)\right) + \lambda_{\max}\left(P^{-1}\left[\left(C(t)\otimes A\right)^T P + P\left(C(t)\otimes A\right)\right]\right)$$
$$+2Lh(t) + h(t)\hat{L} \le \alpha h(t).$$

(6)

2.7 Assumption

There exists a constant $\xi > 1$ such that

$$\ln\left(\xi\beta_k\right) + \alpha\int_{t_{k-1}}^{t_k} h(s)ds < 0, k \in \mathbb{N},$$

(7)

where $\beta_k = \lambda_{\max}\left[P^{-1}\left(E_{nN} + B_k\right)^T P\left(E_{nN} + B_k\right)\right]$.

2.8 Lemma [16]

Let $Y \in \mathbb{R}^{n\times n}$ be a positive definite matrix and $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then, for any $x \in \mathbb{R}^n$, the following inequality holds:

$$\lambda_{\min}\left(Y^1 Q\right)x^T Yx \le x^T Qx \le \lambda_{\max}\left(Y^{-1}Q\right)x^T Yx.$$

(10)

2.9 Lemma (Ito formula)

Let $V(t,X) \in C^{1,2}\left(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}\right)$, then we have Ito formula as follows:

$$dV\left(t,X(t)\right) = LV\left(t,X(t)\right)dt + V_x\left(t,X(t)\right)g\left(t,X(t)\right)dw(t),$$

(8)

where $X(t)$ is the solution of the stochastic differential equation (3) and

$$LV(t,X) = V_x(t,X)f(t,X) + \frac{1}{2}trace\left(g^T(t,X)V_{xx}(t,X)g(t,X)\right).$$

(9)

3 Boundedness Analysis

3.1 Theorem

Assume that Assumptions 2.1–2.3 hold. If $\alpha > 0$, then the impulsive stochastic complex networks in (3) is global mean square asymptotically bounded, and the solution $X(t)$ will eventually converge to the compact set defined by

$$S = \left\{ X \in \mathbf{R}^{nN} \left| E \left\| X\left(t\right) \right\|^2 \le \left(1/\lambda_{\min}\left(P\right)\right)\left(\xi/\xi - 1\right)\left(\left(1/\beta_I\xi\right) - 1\right)\left(\left(2J + \hat{J}\right)\Big/\alpha\right) \right. \right\},$$

where $\beta_I = \inf_{k \in \mathbf{N}} \beta_k$.

Proof. Let the Lyapunov function be in the form of

$$V\left(X\right) = X^T P X. \tag{10}$$

And

$$V_x = X^T P \left[D\left(t\right) X\left(t\right) + \Phi\left(t, X\right) + \left(C\left(t\right) \otimes A\right) X\left(t\right) \right] + \left[D\left(t\right) X\left(t\right) + \Phi\left(t, X\right) \right. \tag{11}$$
$$\left. + \left(C\left(t\right) \otimes A\right) X\left(t\right) \right] P X.$$

Obviously, we have

$$\lambda_{\min}\left(P\right) E \left\| X \right\|^2 \le EV\left(X\right) \le \lambda_{\max}\left(P\right) E \left\| X \right\|^2. \tag{12}$$

According to the Ito formula, the $V\left(X\right)$ along the solution of (3) for $t \in \left(t_{k-1}, t_k\right]$, $k \in \mathbf{N}$, and using Assumptions 2.1 and 2.2 and Lemmas 2.1 and 2.2, we obtain

$$LV\left(t, X\right) = V_x\left(t, X\right) f\left(t, X\right) + \frac{1}{2} trace\left(g^T\left(t, X\right) V_{xx}\left(t, X\right) g\left(t, X\right)\right)$$

$$= X^T P \left[D\left(t\right) X\left(t\right) + \Phi\left(t, X\right) + \left(C\left(t\right) \otimes A\right) X\left(t\right) \right] + \left[D\left(t\right) X\left(t\right) + \Phi\left(t, X\right) \right.$$
$$\left. + \left(C\left(t\right) \otimes A\right) X\left(t\right) \right]^T P X + trace g^T P g$$

$$= X^T\left(t\right) P D\left(t\right) X\left(t\right) + X^T\left(t\right) P \Phi\left(t, X\right) + X^T\left(t\right) P\left(C\left(t\right) \otimes A\right) X\left(t\right)$$
$$+ X^T\left(t\right) D^T\left(t\right) P X\left(t\right) + \Phi^T\left(t, X\right) P X\left(t\right) + X^T\left(t\right)\left(C\left(t\right) \otimes A\right)^T P X\left(t\right)$$
$$+ trace g^T\left(t, X\right) P g\left(t, X\right)$$

$$= X^T\left(t\right)\left(P D\left(t\right) + D^T\left(t\right) P\right) X\left(t\right) + X^T\left(t\right)\left(P\left(C\left(t\right) \otimes A\right) + \left(C\left(t\right) \otimes A\right)^T P\right)$$
$$\times X\left(t\right) + X^T\left(t\right) P \Phi\left(t, X\right) + \Phi^T\left(t, X\right) P X\left(t\right) + trace g^T\left(t, X\right) P g\left(t, X\right)$$

$$\le \left[\lambda_{\max}\left(P^{-1}\left(D^T\left(t\right) P + P D\left(t\right)\right)\right) + \lambda_{\max}\left(P^{-1}\left(\left(C\left(t\right) \otimes A\right)^T P + P\left(C\left(t\right) \otimes A\right)\right)\right) \right.$$
$$+ 2Lh\left(t\right) \Big] X^T\left(t\right) P X\left(t\right) + 2Jh\left(t\right) + h\left(t\right) \hat{L} X^T\left(t\right) P X\left(t\right) + \hat{J} h\left(t\right)$$

$$\le \left[\lambda_{\max}\left(P^{-1}\left(D^T\left(t\right) P + P D\left(t\right)\right)\right) + \lambda_{\max}\left(P^{-1}\left(\left(C\left(t\right) \otimes A\right)^T P + P\left(C\left(t\right) \otimes A\right)\right)\right) \right.$$
$$+ 2Lh\left(t\right) + h\left(t\right) \hat{L} \Big] X^T\left(t\right) P X\left(t\right) + 2Jh\left(t\right) + \hat{J} h\left(t\right) \tag{13}$$

$$\le \alpha h\left(t\right) V\left(t, X\right) + 2Jh\left(t\right) + \hat{J} h\left(t\right) = h\left(t\right) \left[\alpha V\left(t, X\right) + 2J + \hat{J} \right], t \in \left(t_{k-1}, t_k\right]$$

$$\le \alpha h(t) V(t, X) + 2Jh(t) + \hat{J} h(t) = h(t) \left[\alpha V(t, X) + 2J + \hat{J} \right], t \in (t_{k-1}, t_k]. \tag{17}$$

By the variation of the parameter formula, we have

$$EV(t, X(t)) \le EV(t_0, X(t_0)) e^{\alpha \int_{t_0}^{t} h(s)ds} + \left(2J + \hat{J} \right) \int_{t_{k-1}}^{t} h(s) e^{\alpha \int_{s}^{t} h(\zeta)d\zeta} ds,$$

$$t \in (t_{k-1}, t_k], k \in \mathrm{N}. \tag{14}$$

On the other hand, using the discrete part of (3), we have

$$V\left(X\left(t_k^+ \right) \right) = X\left(t_k \right)^T \left[\left(E_{nN} + B_k \right)^T P\left(E_{nN} + B_k \right) \right] X\left(t_k \right)$$

$$\le \lambda_{\max} \left[P^{-1} \left(E_{nN} + B_k \right)^T P\left(E_{nN} + B_k \right) \right] X\left(t_k \right)^T PX\left(t_k \right)$$

$$= \beta_k V\left(X\left(t_k \right) \right), k \in \mathrm{N}. \tag{15}$$

Taking $k = 1$ in the inequality (14), for $t \in (t_0, t_1]$, we get

$$EV(t, X(t)) \le EV(t_0, X(t_0)) e^{\alpha \int_{t_0}^{t} h(s)ds} + \left(2J + \hat{J} \right) \int_{t_0}^{t} h(s) e^{\alpha \int_{s}^{t} h(\zeta)d\zeta} ds. \tag{16}$$

By (15) and(16), we get

$$EV\left(t, X\left(t_1^+ \right) \right) \le \beta_1 EV\left(t_0, X\left(t_0 \right) \right) e^{\alpha \int_{t_0}^{t_1} h(s)ds} + \beta_1 \left(2J + \hat{J} \right) \int_{t_0}^{t_1} h(s) e^{\alpha \int_{s}^{t_1} h(\zeta)d\zeta} ds. \tag{17}$$

Then, for $t \in (t_1, t_2]$, we get

$$EV(t, X(t)) \le \beta_1 EV\left(t_0, X\left(t_0 \right) \right) e^{\alpha \int_{t_0}^{t} h(s)ds}$$

$$+ e^{\alpha \int_{t_1}^{t} h(s)ds} \beta_1 \left(2J + \hat{J} \right) \int_{t_0}^{t_1} h(s) e^{\alpha \int_{s}^{t_1} h(\zeta)d\zeta} ds$$

$$+ \left(2J + \hat{J} \right) \int_{t_1}^{t} h(s) e^{\alpha \int_{s}^{t} h(\zeta)d\zeta} ds. \tag{18}$$

By induction, for $t \in (t_k, t_{k+1}], k \in \mathrm{N}$, we get

$$EV(t, X(t)) \le EV(t, X(t))$$

$$+ \sum_{j=1}^{k} \prod_{i=j}^{k} \beta_i \left(2J + \hat{J} \right) e^{\alpha \int_{t_j}^{t} h(s)ds} \int_{t_{j-1}}^{t_j} h(s) e^{-\alpha \int_{t_j}^{s} h(\zeta)d\zeta} ds$$

$$+ \left(2J + \hat{J} \right) \int_{t_k}^{t} h(s) e^{\alpha \int_{s}^{t} h(\zeta)d\zeta} ds. \prod_{i=1}^{k} \beta_i e^{\alpha \int_{t_0}^{t} h(s)ds}$$

$$\leq EV\left(t,X\left(t\right)\right)\prod_{i=1}^{k}\beta_{i}e^{\alpha\int_{t_{0}}^{t}h(s)ds}$$

$$+\sum_{j=1}^{k}\prod_{i=j}^{k}\beta_{i}\frac{2J+\hat{J}}{\alpha}\left(e^{\alpha\int_{t_{j-1}}^{t}h(s)ds}-e^{\alpha\int_{t_{j}}^{t}h(s)ds}\right)$$

$$+\frac{2J+\hat{J}}{\alpha}e^{\alpha\int_{t_{k}}^{t}h(s)ds}-\frac{2J+\hat{J}}{\alpha}$$

$$\leq EV\left(t_{0},X\left(t_{0}\right)\right)\prod_{i=1}^{k}\beta_{i}e^{\alpha\int_{t_{i-1}}^{t_{i}}h(s)ds}e^{\alpha\int_{t_{k}}^{t}h(s)ds} \quad\quad (19)$$

$$+\sum_{j=1}^{k}\prod_{i=j}^{k}\beta_{i}e^{\alpha\int_{t_{i-1}}^{t_{i}}h(s)ds}e^{\alpha\int_{t_{k}}^{t}h(s)ds}\times\frac{2J+\hat{J}}{\alpha}\left(1-e^{-\alpha\int_{t_{j-1}}^{t_{j}}h(s)ds}\right)$$

$$+\frac{2J+\hat{J}}{\alpha}e^{\alpha\int_{t_{k}}^{t}h(s)ds}-\frac{2J+\hat{J}}{\alpha}.$$

On the other hand, it follows from (7) that

$$\beta_{k}e^{\alpha\int_{t_{k-1}}^{t_{k}}h(s)ds}<\frac{1}{\xi},\text{ or }e^{\alpha\int_{t_{k-1}}^{t_{k}}h(s)ds}<\frac{1}{\beta_{k}\xi},k\in\text{N}. \quad\quad (24)$$

For $t\in\left(t_{k},t_{k+1}\right],k\in\text{N}$, by $\alpha>0$, (19), and**Error! Reference source not found.**, we have

$$EV\left(t,X\left(t\right)\right)\leq EV\left(t_{0},X\left(t_{0}\right)\right)\left(\frac{1}{\xi}\right)^{k}\frac{1}{\beta_{k+1}\xi}$$

$$+\sum_{j=1}^{k}\left(\frac{1}{\xi}\right)^{k-j+1}\frac{1}{\beta_{k+1}\xi}\frac{2J+\hat{J}}{\alpha}\left(1-\beta_{j}\xi\right)+\frac{2J+\hat{J}}{\alpha}\left(\frac{1}{\beta_{k+1}\xi}-1\right) \quad\quad (25)$$

$$\leq EV\left(t_{0},X\left(t_{0}\right)\right)\left(\frac{1}{\xi}\right)^{k}\frac{1}{\beta_{I}\xi}+\frac{\xi}{\xi-1}\left(\frac{1}{\beta_{I}\xi}-1\right)\frac{2J+\hat{J}}{\alpha}.$$

By (12) and **Error! Reference source not found.**, for $t\in\left(t_{k},t_{k+1}\right],k\in\text{N}$, we get

$$E\left\|X\left(t\right)\right\|^{2}\leq\frac{\lambda_{\max}\left(P\right)}{\lambda_{\min}\left(P\right)}E\left\|X\left(t_{0}\right)\right\|\left(\frac{1}{\xi}\right)^{k}\frac{1}{\beta_{I}\xi}+M, \quad\quad (20)$$

where $M=\left(1/\lambda_{\min}\left(P\right)\right)\left(\xi/\xi-1\right)\left(\left(1/\beta_{I}\xi\right)-1\right)\left(\left(2J+\hat{J}\right)/\alpha\right)$. Using (20), and noting $\xi>1$ and

$\lim_{k\to\infty} t_k = \infty$, we get $\lim_{t\to\infty} E\|X(t)\|^2 \le (1/\lambda_{\min}(P))(\xi/\xi-1)((1/\beta_I\xi)-1)\left(\left(2J+\hat{J}\right)\Big/\alpha\right)$.

The proof is completed.

3.2 Theorem

Assume that Assumptions 2.1–2.3 hold. If $\alpha < 0$, then the impulsive stochastic complex networks in (3) is global mean square asymptotically bounded, and the solution $X(t)$ will eventually converge to the compact set defined be $S = \left\{ X \in \mathrm{R}^{nN} \,\middle|\, E\|X(t)\|^2 \le M \right\}$,where

$$M = (1/\lambda_{\min})\left(e^{-\alpha Hs} - 1\right)\left((1/(\xi-1)) + e^{\alpha Hs}\right) \times \left(\left(2J+\hat{J}\right)\Big/-\alpha\right), \quad Hs = \sup_{k\in\mathrm{N}} \int_{t_{k-1}}^{t_k} h(s)ds < \infty .$$

Proof. It follows from (19), **Error! Reference source not found.**, and $\alpha < 0$ that

$$EV(t, X(t)) \le EV(t_0, X(t_0))\left(\frac{1}{\xi}\right)^k + \sum_{j=1}^{k}\prod_{i=j}^{k}\beta_i e^{\alpha\int_{t_{i-1}}^{t_i} h(s)ds}$$

$$\times \frac{2J+\hat{J}}{-\alpha}\left(e^{-\alpha\int_{t_{j-1}}^{t_j} h(s)ds} - 1\right) + \frac{2J+\hat{J}}{-\alpha}\left(1 - e^{\alpha\int_{t_k}^{t} h(s)ds}\right)$$

$$\le EV(t_0, X(t_0))\left(\frac{1}{\xi}\right)^k + \sum_{j=1}^{k}\left(\frac{1}{\xi}\right)^{k-j+1}\frac{2J+\hat{J}}{-\alpha}\left(e^{-\alpha Hs} - 1\right) + \frac{2J+\hat{J}}{-\alpha}\left(1 - e^{\alpha Hs}\right) \tag{27}$$

$$\le EV(t_0, X(t_0))\left(\frac{1}{\xi}\right)^k + \frac{2J+\hat{J}}{-\alpha}\left(e^{-\alpha Hs} - 1\right)\left(\frac{1}{\xi-1} + e^{\alpha Hs}\right),$$

$t \in (t_k, t_{k+1}], k \in \mathrm{N}.$

By (12) and **Error! Reference source not found.**, for $t \in (t_k, t_{k+1}], k \in \mathrm{N}$, we get

$$E\|X\|^2 \le \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} E\|X(t_0)\|\left(\frac{1}{\xi}\right)^k + M. \tag{21}$$

Using (21), and noting $\xi > 1$ and $\lim_{k\to\infty} t_k = \infty$,we get $\lim_{t\to\infty} E\|X(t)\|^2 \le M$ The proof is completed.

3.3 Theorem

Assume that Assumptions 2.1–2.3 hold. If $\alpha = 0$, then the impulsive stochastic complex networks in (3) is global mean square asymptotically bounded, and the solution $X(t)$ will eventually converge to the

compact set defined be $S = \left\{ X \in \mathrm{R}^{nN} \left| E \left\| X\left(t\right) \right\|^2 \leq 1 \middle/ \lambda_{\min}\left(P\right)\left(\xi/\left(\xi-1\right)\right)\left(2J+\hat{J}\right)Hs \right. \right\}$, where

$$Hs = \sup_{k \in \mathrm{N}} \int_{t_{k-1}}^{t_k} h\left(s\right) \times ds < \infty .$$

Proof. By (19) and $\alpha = 0$, we have

$$EV\left(t, X\left(t\right)\right) \leq EV\left(t, X\left(t\right)\right) \prod_{i=1}^{k} \beta_i$$

$$+ \sum_{j=1}^{k} \prod_{i=j}^{k} \beta_i \left(2J+\hat{J}\right) \int_{t_{j-1}}^{t_j} h\left(s\right)ds$$

$$+ \left(2J+\hat{J}\right) \int_{t_k}^{t} h\left(s\right)ds, \tag{22}$$

$$t \in \left(t_k, t_{k+1}\right], k \in \mathrm{N}.$$

On the other hand, from (7) with $\alpha = 0$, we have

$$\beta_k < 1/\xi . \tag{23}$$

From (22) and(23), it follows that for $t \in \left(t_k, t_{k+1}\right], k \in \mathrm{N}$, we get

$$EV\left(t, X\left(t\right)\right) \leq EV\left(t, X\left(t\right)\right)\left(\frac{1}{\xi}\right)^k + \sum_{j=1}^{k} \left(\frac{1}{\xi}\right)^{k-j+1} \left(2J+\hat{J}\right)Hs + \left(2J+\hat{J}\right)Hs$$

$$\leq EV\left(t, X\left(t\right)\right)\left(\frac{1}{\xi}\right)^k + \left(\frac{\xi}{\xi-1}\right)\left(2J+\hat{J}\right)Hs. \tag{24}$$

By (12) and(24), for $t \in \left(t_k, t_{k+1}\right], k \in \mathrm{N}$, we get

$$E\left\|X\left(t\right)\right\|^2 \leq \frac{\lambda_{\max}\left(P\right)}{\lambda_{\min}\left(P\right)} E\left\|X\left(t_0\right)\right\|\left(\frac{1}{\xi}\right)^k + \frac{1}{\lambda_{\min}\left(P\right)}\left(\frac{\xi}{\xi-1}\right)\left(2J+\hat{J}\right)Hs. \tag{25}$$

Using(25), and noting $\xi > 1$ and $\lim_{k \to \infty} t_k = \infty$,we get $\lim_{t \to \infty} E\left\|X\left(t\right)\right\|^2 \leq 1/\lambda_{\min}\left(P\right)\left(\xi/\left(\xi-1\right)\right)$

$\times \left(2J+\hat{J}\right)Hs$. The proof is completed.

The following corollaries follow directly from Theorems 3.1–3.3. Here, we omit their proofs to avoid repetition.

3.4 Corollary

If Assumptions 2.1–2.3 with $J = 0$ hold, then the impulsive stochastic complex networks in (3) is global mean square asymptotically stable.

3.5 Corollary

If Assumptions 2.1–2.3 with $h(t) \equiv 1$ and $P = E_{nN}$ hold, then the impulsive stochastic complex networks in (3) is global mean square asymptotically bounded, and the solution $X(t)$ will eventually converge to the compact set defined by $S = \left\{ X \in \mathrm{R}^{nN} \left| E \left\| X(t) \right\|^2 \leq M \right. \right\}$, where

$$
M = \begin{cases}
\dfrac{\xi}{\xi - 1} \left(\dfrac{1}{\beta_1 \xi} - 1 \right) \dfrac{2J + \hat{J}}{\alpha}, \alpha > 0 \\[3mm]
\left(e^{-\alpha Hs} - 1 \right) \left(\dfrac{1}{\xi - 1} + e^{\alpha Hs} \right) \dfrac{2J + \hat{J}}{-\alpha}, \alpha < 0, \\[3mm]
\dfrac{\xi}{\xi - 1} \left(2J + \hat{J} \right) Hs, \alpha = 0
\end{cases}
\tag{26}
$$

where $\beta_1 = \inf_{k \in \mathrm{N}} \left\{ \lambda_{\max} \left[P^{-1} \left(E_{nN} + B_k \right)^T P \left(E_{nN} + B_k \right) \right] \right\}$, and $Hs = \sup_{k \in \mathrm{N}} \left\{ t_k - t_{k-1} \right\} < \infty$.

3.6 Remark

It should be noted that stochastic effects are not considered in [16]. So the present result is applied more widely in complex networks compare to the one in [16].

4 Conclusion

The mean square asymptotic boundedness of a class of stochastic complex systems with different dynamic nodes has been investigated in this paper. Using the Lyapunov function and Ito formula, the mean square asymptotic boundedness and mean square asymptotic stability conditions of stochastic complex systems with different dynamic nodes have been obtained. How to extend the current results to the delay case is still a challenging problem and need further study in the future work.

Competing Interests

Authors have declared that no competing interests exist.

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