On A Class of Idealized Near-Rings Admitting Frobenius Derivations

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Authors' contributions

This work was carried out in collaboration among all the authors. OMO designed the study, wrote both the protocol and the first draft of the manuscript. OBA and AJM performed the mathematical analysis of the study and sharpened some of the results. A.J.M managed the literature searches. All the authors read and approved the final manuscript.

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Abstract

In this paper, we use the idealization procedure for finite rings to construct a class of quasi-3 prime Near-Rings $\mathcal{N}$ with a Jordan ideal $J(\mathcal{N})$ and admitting a Frobenius derivation. The structural characterization of $\mathcal{N}$, $J(\mathcal{N})$ and commutation of $\mathcal{N}$ via the Frobenius derivations have been explicitly determined.

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1   Introduction

The process of idealization of finite local rings by adjoining the Galois subring R_0 with its maximal submodules has contributed immensely to the novel approach of classification of finite rings. Indeed a number of researchers have obtained results demonstrating various characterizations of finite local rings with $n-$nilpotent radicals of Jacobson, for some values of n using certain well known structures. For detailed studies regarding such classes of rings, reference can be made to ([1, 2] and the references there in). Despite the fact that, all rings are well known to be near-rings [3], the idealization concept has not been extrapolated to give dual results concerning near-rings; algebraic structures that lie between rings and non-rings and hence useful in non-commutative Ring Theory.

A number of attempts have been made to study the properties of near rings. For instance, the commutativity properties have been investigated using certain maps called derivations. The pioneer work on derivations on near rings was conducted by Bell and Mason [4] where they characterized the derivations on near rings and near fields. Their work was however motivated by the study by Posner [5] concerning derivations on prime rings, mappings that did not have suitable constraints and also not generalized for Near-Rings. A characterization of the commutativity property of prime and semi-prime rings having certain constraints on derivation has been advanced by a number of other algebraists, see for example ([6] and references there in). Some similar results have also bee obtained for Near-Rings (cf.[4, 7, 8]). Kamal and Khalid [9] in their study of commutativity of near rings with derivations found that any Near-ring admits a derivation if and only if it is zero-symmetric. They also proved some commutativity Theorems for a non-necessarily 3-prime Near-Rings with a suitably constrained derivation d, with the condition that $d(a)$ is not a left zero divisor in R, for some $a \in R$. As a consequence, they attempted to advance further research around the classification of 3-prime Near-Rings admitting derivations.

The most recent works of Asma and Inzamam [10] on Commutativity of a 3-Prime near Ring satisfying certain differential identities on Jordan Ideals has given an investigation of Near-Rings admitting derivation a new twist. In fact, they proposed a number of necessary results that qualify 3-prime near rings with Jordan ideals admitting derivations and multiplicative derivations to be commutative. In particular, they proved the commutativity condition for a 3-prime near ring $\mathcal{N}$ under any one of the following conditions: (i) $[d_1(u), d_2(k)] = [u, k]$, (ii) $d([k, u]) = [d(k), u]$, (iii) $[d(u), k] = [u, k]$,(iv) $d([k, u]) = d(k) \circ u$, (v) $[d(k), d(u)] = 0$ for all $u \in \mathcal{N}$ and $k \in J$, a Jordan ideal of $\mathcal{N}$, where d, d_1, d_2 are derivations on $\mathcal{N}$. Bell and Daif [11] showed the following result: If R is a 2-torsion free prime ring admitting a strong commutativity preserving derivation d, that is d satisfies $[d(v), d(w)] = [v, w]$ for every $v, w \in R$, then R is commutative. This result has been extended by Asma and Inzamam [10] for a 3-prime near ring in two directions. First of all, they considered two derivations instead of one derivation, and secondly, they proved the commutativity of a 3-prime near ring N in place of a ring R in case of a Jordan ideal of N. These results provided very good basic necessary conditions for studying other classes of Near-Rings, including the idealized ones. Perhaps based on the recommendations of [10], it would be interesting to investigate whether their results would still hold when another type of derivation is used on general near rings and idealized near rings. On the other hand, little is documented about any class of idealized Near-Rings. Therefore, in the current paper, we construct and characterize a class idealized near rings whose commutation is investigated using a new class of differential identities called the Frobenius derivations.

2   Some Important Definitions

Definition 2.1. *An algebraic structure $\mathcal{N}$ endowed with two binary operations $+$ and $\cdot$ such that (i) $(\mathcal{N}, +)$ is a group, (ii) $(\mathcal{N}, \cdot)$ is a semigroup, and (iii) $u \cdot (v + w) = u \cdot v + u \cdot w$ for every $u, v, w \in \mathcal{N}$ is called a left Near-Ring. Dually, if instead of (iii), $\mathcal{N}$ satisfies the right distributive*

law, then $\mathcal{N}$ is said to be a right near ring. Therefore, near rings are generalized rings, need not be commutative, and most importantly, only one distributive law is postulated (Pilz [3]).

Definition 2.2. *A near ring $\mathcal{N}$ is known as zero-symmetric if $0 \cdot u = 0$ for every $u \in \mathcal{N}$. Let $\mathcal{N}$ be a zero-symmetric left near ring with $Z(\mathcal{N})$ as its multiplicative center, if $2u = 0 \Leftrightarrow u = 0$, then $\mathcal{N}$ is called 2-torsion free. Otherwise, if $2u = 0$ and $u \neq 0$, we shall call $\mathcal{N}$ non-2-torsion free.*

Definition 2.3. *For $v, w \in \mathcal{N}$, where $\mathcal{N}$ is a near-ring, the symbols $[v, w]$ and $v \circ w$ denote the commutator $vw - wv$ and the anticommutator $vw + wv$, respectively.*

Definition 2.4. *An additive subgroup $\mathcal{J}$ of a near ring $\mathcal{N}$ is known as a Jordan ideal of $\mathcal{N}$ if $k \circ u \in \mathcal{J}$ and $u \circ k \in \mathcal{J}$ for all $k \in \mathcal{J}$ and $u \in \mathcal{N}$.*

Definition 2.5. *A mapping $d : R \longrightarrow R$ is known as a multiplicative derivation on a ring R if $d(vw) = d(v)w + vd(w)$ for every $v, w \in R$. Thus, a mapping $d : \mathcal{N} \longrightarrow \mathcal{N}$ is known as a multiplicative derivation on a near ring $\mathcal{N}$ if $d(vw) = d(v)w + vd(w)$ for every $v, w \in N$.*

3   The Construction of Idealized Near Ring

Let $R_0 = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k and let $\mathcal{M} = \langle u_i \rangle : i = 1, \cdots h$ be an $h-$dimensional near-module of R_0 so that the ordered pair $(\mathcal{N}, +) = (R_0 \oplus \mathcal{M}, +)$ is a group. On $\mathcal{N}$, let $p^k u_i = \prod_{i=1}^{k} u_i = 0$ and $u_i r_0 = (r_0)^{d_i} u_i$ when $k = 1$, 2 where $r_0 \in R_0$, k, r are invariants and d_i a Frobenius derivation associated with elements of $\mathcal{M}$ and given by; $d_i(u_i) = (u_i)^p$. Let $\mathcal{J}$ be a near-ideal of $\mathcal{M}$ satisfying the condition that whenever $u_i, u_j \in \mathcal{J}$, we have $u_i \circ u_j \in \mathcal{J}$ or $u_i \circ u_j = 0$. If λ_i are any units of R_0, then we can see that the elements of $\mathcal{N} = R_0 \oplus \mathcal{M}$ are of the form: $x = r_0 + \sum_{i=1}^{h} \lambda_i u_i$. In fact, if $x = r_0 + \sum_{i=1}^{h} \alpha_i u_i$ and $y = s_0 + \sum_{i=1}^{h} \beta_i u_i$ are any two elements of $\mathcal{N}$, then we have their product as:

$$x \cdot y = \left(r_0 + \sum_{i=1}^{h} \alpha_i u_i\right) \cdot \left(s_0 + \sum_{i=1}^{h} \beta_i u_i\right) = r_0 s_0 + \sum_{i=1}^{h} \left\{\beta_i(r_0 + p^k R_0)^{d_i} + \alpha_i(s_0 + p^k R_0)^{d_i}\right\} u_i.$$

Theorem 3.1. *The triplet $(\mathcal{N}, +, \cdot)$ with the product given in the construction is a left(respective right) Near-Ring whose unique maximal ideal is the Jordan ideal $\mathcal{J}(\mathcal{N})$.*

Proof. Since $(\mathcal{N}, +)$ is a group, we only show that $(\mathcal{N}, \cdot)$ is a semi group and that the left(right) distributive law holds on $\mathcal{N}$

Let $x, y, z \in \mathcal{N}$ be defined by: $x = r_0 + \sum_{i=1}^{h} \alpha_i u_i$, $y = s_0 + \sum_{i=1}^{h} \beta_i u_i$, $z = k_0 + \sum_{i=1}^{h} \gamma_i u_i$ where $\alpha, \beta, \gamma \in R_0^*$ or $\alpha_i, \beta_i, \gamma_i \in p^k R_0$, then,

$$\left(r_0 + \sum_{i=1}^{h} \alpha_i u_i\right) \left\{(s_0 + \sum_{i=1}^{h} \beta_i u_i)(k_0 + \sum_{i=1}^{h} \gamma_i u_i)\right\}$$

$$= \left(r_0 + \sum_{i=1}^{h} \alpha_i u_i\right) \left(\left\{s_0 k_0 + \sum_{i=1}^{h} \left(\gamma_i(s_0 + pR_0) + \beta_i(k_0 + pR_0)^{d_i}\right)\right\} u_i\right)$$

$$= r_0 s_0 k_0 + r_0 \left(\sum_{i=1}^{h} \left\{\gamma_i(s_0 + pR_0) + \beta_i(k_0 + pR_0)^{d_i}\right\} u_i\right) +$$

$$\sum_{i=1}^{h} \alpha_i u_i \left(s_0 k_0 + \sum_{i=1}^{h} \left\{\gamma_i(s_0 + pR_0) + \beta_i(k_0 + pR_0)^{d_i}\right\}\right)$$

$$= r_0 s_0 k_0 + \sum_{i=1}^{h} \Big\{ \gamma_i (r_0 + pR_0) + \alpha_i (s_0 + pR_0)^{d_i} + \beta_i (k_0 + pR_0)^{d_i} \Big\} u_i$$

$$= r_0 s_0 + \sum_{i=1}^{h} \Big\{ \beta_i (r_0 + pR_0) + \alpha_i (s_0 + pR_0)^{d_i} \Big\} u_i (k_0 + \sum_{i=1}^{h} (\gamma_i u_i).$$

$$= \Big(\Big\{ (r_0 + \sum_{i=1}^{h} \alpha_i u_i)(s_0 + \sum_{i=1}^{h} \beta_i u_i) \Big\} \Big) \Big(k_0 + \sum_{i=1}^{h} \gamma_i u_i \Big).$$

Thus, the multiplication given by the construction is associative.

Next,

$$\Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i \Big) \Big\{ (s_0 + \sum_{i=1}^{h} \beta_i u_i) + (k_0 + \sum_{i=1}^{h} \gamma_i u_i) \Big\}$$

$$= \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i \Big) \Big\{ (s_0 + k_0) + \sum_{i=1}^{h} (\beta_i + \gamma_i) u_i \Big\}$$

$$= r_0 s_0 + r_0 k_0 + \sum_{i=1}^{h} \Big\{ (r_o + pR_0)\beta_i + \alpha_i (s_0 + pR_0)^{d_i} \Big\} u_i + \sum_{i=1}^{h} \Big\{ (r_o + pR_0)\gamma_i + \alpha_i (k_0 + pR_0)^{d_i} \Big\} u_i$$

$$= r_0 s_0 + \sum_{i=1}^{h} \Big\{ (r_0 + pR_0)\beta_i + \alpha_i (s_0 + pR_0)^{d_i} \Big\} u_i + r_0 k_0 + \sum_{i=1}^{h} \Big\{ (r_0 + pR_0)\gamma_i + \alpha_i (k_0 + pR_0)^{d_i} \Big\} u_i$$

$$= \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i \Big) \cdot \Big(s_0 + \sum_{i=1}^{h} \beta_i u_i \Big) + \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i \Big) \cdot \Big(k_0 + \sum_{i=1}^{h} \gamma_i u_i \Big),$$

the desired left distributive law. $\square$

Remark 3.1. *Whenever $d_i = i_{\mathcal{N}}$ the identity map, then, $\mathcal{N} = R \oplus \mathcal{M}$ is a commutative Near-Ring with identity $(1, 0, ..., 0)$*

Proposition 3.1. *The Frobenius derivation $d : \mathcal{N} \to \mathcal{N}$ is an endomorphism whenever $u^{1-p} + v^{1-p} = 1$ for any $u, v \in \mathcal{N}$.*

Proof. Let $\mathcal{N}$ constructed be a Near-Ring with unity, then by Bezout's Theorem, for some non-zero divisors $u, v \in \mathcal{N}$, $u^{1-p} + v^{1-p} = 1$ holds. Now, from the construction, $R_0 = GR(p^k, p^k)$ is a maximal subset of $\mathcal{N}$, the characteristic of R_0 coincides with the characteristic of $\mathcal{N}$, thus $p^k u = 0$, $k = 1, 2 \forall u \in \mathcal{N}$. Let $u = r_0 + \sum_{i=1}^{h} \alpha_i u_i$, and $v = s_0 + \sum_{i=1}^{h} \beta_i u_i$, then clearly u, v are in $\mathcal{N}$ and by definition of our derivation; $d(u) = u^p$. But,

$$d(uv) = d(u)v + ud(v) = u^p v + uv^p = u^p (v + u^{1-p} v^p) = u^p (\underbrace{v^{1-p} + u^{1-p}}_{1}) v^p = u^p v^p$$

Thus,

$$d(uv) = u^p v^p = d(u)d(v).$$

Next,

$$d(u + v) = (u + v)^p = \sum_{i=1}^{p} \binom{p}{i} u^{p-i} v^i.$$

$$= \begin{pmatrix} p \\ 0 \end{pmatrix} u^p v^0 + \begin{pmatrix} p \\ 1 \end{pmatrix} u^{p-1} v + \cdots + \begin{pmatrix} p \\ p \end{pmatrix} u^0 v^p$$

$$= u^p + \underbrace{p u^{p-1} v + \cdots}_{0} + v^p = u^p + v^p = d(u) + d(v).$$

$\square$

The next result presents a characterization of the direct products of classes of the near-rings constructed via matrix ring type near-rings

Theorem 3.2. *Let* $\{\mathcal{N}_i\} = \{\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_h\}$ *be a family of classes of the near-rings constructed and define* $\mathcal{N}_1 = R_0 \oplus \mathcal{M}_1$, $\mathcal{N}_2 = R_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$, $\cdots, \mathcal{N}_h = R_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_h$ *where* $\mathcal{M}_1 = < u_1 >, \mathcal{M}_2 = < u_1, u_2 >, \cdots, \mathcal{M}_h = < u_1, \cdots, u_h >$ *and* $\mathcal{T} = \prod_{i=1}^h \mathcal{N}_i$ *be their direct products. Then for any* $n \geq 1$*, the rings* $M_n(\mathcal{T})$ *and* $S = \prod_{i=1}^h M_n(\mathcal{N}_i)$ *are isomorphic.*

Proof. Let $n \geq 1$, $\Theta = M_n(\mathcal{T})$, $\Omega = \prod_{i=1}^h M_n(\mathcal{N}_i)$. For any $A = [a_{rs}]_{n \times n} \in \Theta$, let $a_{rs} = \prod_{i=1}^h a_{irs} \in \mathcal{T}$ for any $r, s \in \{1, \cdots, n\}$. Now, for each $i \in \{1, \cdots, h\}$, let $A_i = [a_{rs}]_{n \times n} \in M_n(\mathcal{N}_i)$, then, it is easy to verify that the map $f : \Theta \to \Omega$ with $f(A) = \prod_{i=1}^h A_i$ is an additive group isomorphism. To see that f is indeed a NR homomorphism: let $B = [b_{rs}]_{n \times n} \in \Theta$ and set $AB = C$, then $C = [c_{rs}]_{n \times n} \in \Theta$ where,

$$c_{rs} = \sum_{t=1}^n \Big(\prod_{i=1}^h a_{irt} \Big) \Big(\prod_{i=1}^h b_{irt} \Big) = \prod_{i=1}^h \Big(\sum_{t=1}^n a_{irt} b_{its} \Big) \in \mathcal{T}.$$

Hence, $c_{irs} = \sum_{t=1}^n a_{irt} b_{its}$, $: 1 \leq i \leq h$, $r, s \in \{1, \cdots, n\}$. Thus, by definition, $f(C) = \prod_i C_i$ where $C_i = [c_{irs}]_{n \times n} = [\sum_{t=1}^n a_{irt} b_{its}]_{n \times n} \in M_n(\mathcal{N}_i)$. Dually,

$$f(A)f(B) = \prod_{i=1}^h (A_i B_i) = \prod_i \Big([a_{irs}][b_{its}] \Big) = \prod_i \Big[\sum_{t=1}^n a_{irs} b_{its} \Big] \in \Omega.$$

Thus, $f(AB) = f(A)f(B)$, as required. $\square$

4 Jordan Ideal and Commutation of $\mathcal{N}$ via the Frobenius Derivation

We investigate the commutation over the constructed $\mathcal{N}$ using the properties of the Jordan ideal $J(\mathcal{N})$ and the Frobenius derivations d, d_1 and d_2 admitted by $\mathcal{N}$. In the sequel, the following results hold:

Theorem 4.1. *The Jordan ideal of* $\mathcal{N}$ *denoted by* $J(\mathcal{N})$ *is of the form:* $J(\mathcal{N}) = (0) \oplus \sum_{i=1}^h \alpha_i u_i$ *and* $\Big(J(\mathcal{N}) \Big)^2 = (0)$*, so the near-ring constructed has a 2-nilpotent radical of Jordan ideal.*

Proof. Suppose $u \in \mathcal{N}$ with $u = r_0 + \sum_{i=1}^h \alpha_i u_i$ and from the construction of $\mathcal{N}$, we have that $p^k u_i = 0$ for any prime p with $k = 1, 2$ so $2u = 2(r_0 + \sum_{i=1}^h \alpha_i u_i) = 0$, implies that $u = r_0 + \sum_{i=1}^h \alpha_i u_i \neq 0$, necessarily, thus $\mathcal{N}$ is zero-symmetric but non-2-torsion free. Now, let $d : \mathcal{N} \to \mathcal{N}$ be an identity Frobenius derivation obeying the product on $\mathcal{N}$, then the anti-commutator of u and itself is $u \circ u$ and given by:

$$\Big(r_0 + \sum_{i=1}^h \alpha_i u_i \Big) \circ \Big(r_0 + \sum_{i=1}^h \alpha_i u_i \Big) = \Big\{ \Big(r_0 + \sum_{i=1}^h \alpha_i u_i \Big) \Big(r_0 + \sum_{i=1}^h \alpha_i u_i \Big) \Big\} + \Big\{ \Big(r_0 + \sum_{i=1}^h \alpha_i u_i \Big) \Big(r_0 + \sum_{i=1}^h \alpha_i u_i \Big) \Big\}$$

$$= 2\Big\{\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big\}$$

$$= 2\Big(r_0^2 + \underbrace{\sum_{i=1}^h 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i}_{\in \mathcal{N}}\Big) = 0 \in J(\mathcal{N}).$$

Thus $J(\mathcal{N})$ qualifies to be the Jordan ideal of $\mathcal{N}$ by definition. Next, since the $\mathrm{char}R_0 = \mathrm{char}\mathcal{N} = p^k; k = 1, 2$, it is immediate that

$$2\Big(r_0^2 + \underbrace{\sum_{i=1}^h 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i}_{\in \mathcal{N}}\Big) = 2r_0^2 + 2\sum_{i=1}^h 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i = 0 \oplus \sum_{i=1}^h \alpha_i u_i \in J(\mathcal{N}).$$

So

$$J(\mathcal{N}) \cong (0) \oplus \sum_{i=1}^h \alpha_i u_i.$$

Finally, since $2\Big\{\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big\}$ is in $J(\mathcal{N})$,

$$\Big(2\Big\{\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big\}\Big)^2 = \Big(2\Big(r_0^2 + \underbrace{\sum_{i=1}^h 2\Big((r_0 + pR_0)^d\Big)\alpha_i u_i}_{\in \mathcal{N}}\Big)\Big)^2 = 0.$$

Hence the required condition

$$\Big(J(\mathcal{N})\Big)^2 \cong (0).$$

$\square$

Claim 4.1. *Let $\mathcal{N} = R_0 \oplus \mathcal{M}$ be the near-ring constructed. Since $p^k u_i = \prod_{i=1}^h u_i = 0; k = 1, 2$, it means that if $k = 2$, and $R_0 = \{0\}$ then $u^2 = 0$ so that $u\mathcal{N}u = 0$. So, $\mathcal{N}$ is said to be quasi-3 prime near ring of characteristic p or p^2.*

Next, we investigate some commutativity properties of $\mathcal{N}$.

Proposition 4.1. *Let $\mathcal{N}$ be the near-ring of the construction and $J(\mathcal{N})$ be its Jordan ideal. Then $J(\mathcal{N}) = \Big\{0 + \sum_{i=1}^h \alpha_i u_i\Big\} \subseteq Z(\mathcal{N})$.*

Proof. By definition of center, we have that for all $\Big(s_0 + \sum_{i=1}^h \beta_i u_i\Big) \in \mathcal{N}$

$$Z(\mathcal{N}) = \Big\{r_0 + \sum_{i=1}^h \alpha_i u_i \big| \Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big(s_0 + \sum_{i=1}^h \beta_i u_i\Big) = \Big(s_0 + \sum_{i=1}^h \beta_i u_i\Big)\Big(r_0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big\}.$$

If $J(\mathcal{N}) = (0)$ we are done because, trivially $0 \in Z(\mathcal{N})$ and thus $J(\mathcal{N}) \subseteq Z(\mathcal{N})$.

Otherwise, let $w \in J(\mathcal{N})$ and $v \in \mathcal{N}$, with $w = 0 + \sum_{i=1}^h \alpha_i u_i$ and $v = s_0 + \sum_{i=1}^h \beta_i u_i$, then

$$wv = \Big(0 + \sum_{i=1}^h \alpha_i u_i\Big)\Big(s_0 + \sum_{i=1}^h \beta_i u_i\Big) = 0 + \sum_{i=1}^h \Big\{\beta_i(0 + p^k R_0)^{d_i} + \alpha_i(s_0 + p^k R_0)^{d_i}\Big\}u_i \in J(\mathcal{N}),$$

and

$$vw = \Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big)\Big(0 + \sum_{i=1}^{h} \alpha_i u_i\Big) = 0 + \sum_{i=1}^{h} \Big\{\alpha_i(s_0 + p^k R_0)^{d_i} + \beta_i(0 + p^k R_0)^{d_i}\Big\}u_i \in J(\mathcal{N}).$$

Since the usual addition is commutative on $\mathcal{N}$, we see that $wv = vw \in J(\mathcal{N})$ and thus $w = \Big(0 + \sum_{i=1}^{h} \alpha_i u_i\Big) \in J(\mathcal{N})$ is also a member of $Z(\mathcal{N})$, which clears the proof. Thus,

$$J(\mathcal{N}) \subseteq Z(\mathcal{N}).$$

$\square$

Claim 4.2. *Given that $J(\mathcal{N}) \subseteq Z(\mathcal{N})$, $\mathcal{N}$ is a commutative near-ring.*

Theorem 4.2. *(cf. [12]) Let $J(\mathcal{N}) \neq (0)$ be the Jordan ideal of the non-2-torsion free quasi 3-prime near-ring $\mathcal{N}$. Let $d : \mathcal{N} \to \mathcal{N}$ be the multiplicative Frobenius derivation of the construction such that $d(J(\mathcal{N})) = 0$, then either $d = 0$ or $\Big\{0 + \sum_{i=1}^{h} \alpha_i u_i\Big\} \subseteq Z(\mathcal{N})$.*

Proof. If $d = 0$, we are done.

If $d \neq 0$, then the elements of $J(\mathcal{N})$ commute under the multiplication on $\mathcal{N}$. Therefore,

$$Z(J(\mathcal{N})) \cap Z(\mathcal{N}) = \Big\{0 + \sum_{i=1}^{h} \alpha_i u_i\Big\}.$$

So $0 + \sum_{i=1}^{h} \alpha_i u_i \in Z(\mathcal{N})$ and $\Big\{0 + \sum_{i=1}^{h} \alpha_i u_i\Big\} \subseteq Z(\mathcal{N})$. $\square$

Theorem 4.3. *Let $J(\mathcal{N}) \neq (0)$ be the Jordan ideal of the non-2-torsion free quasi 3-prime near-ring $\mathcal{N} = R_0 \oplus \mathcal{M}$. Let $d : \mathcal{N} \to \mathcal{N}$ be the multiplicative Frobenius derivation of the construction. If $u, v, w \in \mathcal{N}$ and $r_0, s_0, k_0 \in R_0$, $\Big\{(r_0 + pR_0)\gamma_i + (s_0 + pR_0)^d \beta_i + (k_0 + pR_0)^d \alpha_i\Big\} \subseteq Ann(\mathcal{M})$, then $d(uvw) = d(r_0, s_0, k_0)$. Moreover,*

$$\Big(d(uv)\Big)w = \Big(d(u)v + ud(v)\Big)w = d(u)vw + ud(v)w.$$

Proof. With obvious identifications,

$$u = r_0 + \sum_{i=1}^{h} \alpha_i u_i, \ v = s_0 + \sum_{i=1}^{h} \beta_i u_i, \ w = k_0 + \sum_{i=1}^{h} \gamma_i u_i$$

so,

$$d(uvw) = d\Big(r_0 s_0 k_0 + \sum_{i=1}^{h} \Big\{\underbrace{(r_0 + pR_0)\gamma_i + (s_0 + pR_0)^d \beta_i + (k_0 + pR_0)^d \alpha_i}_{\sigma}\Big\}u_i\Big)$$

$$= d\Big(r_0 s_0 k_0 + \sigma u_i\Big) = \Big(r_0 s_0 k_0 + \sigma u_i\Big)^p = \Big(r_0 s_0 k_0\Big)^p = d\Big(r_0 s_0 k_0\Big).$$

Next, **case (i)**

$$d(uvw) = d(uv)w + uvd(w) = \Big(d(u)v + ud(v)\Big)w + uvd(w)$$

$$= \Big(u^p v + uv^p\Big)w + uvw^p = u^p vw + uv^p w + uvw^p.$$

Also, **case (ii)**

$$d(uvw) = d(u)vw + ud(vw) = d(u)vw + u\Big(d(v)w + vd(w)\Big)$$

$$= d(u)vw + ud(v)w + uvd(w) = u^p vw + uv^p w + uvw^p$$

From, (i) and (ii), it follows that $\Big(d(uv)\Big)w = \Big(d(u)v + ud(v)\Big)w = d(u)vw + ud(v)w$. $\square$

Theorem 4.4. *Let $J(\mathcal{N})$ be the non-zero Jordan ideal of $\mathcal{N}$. Suppose $d_1 : \mathcal{N} \to \mathcal{N}$ and $d_2 : \mathcal{N} \to J(\mathcal{N})$ are two non-zero Frobenius derivations such that d_2 commutes on $J(\mathcal{N})$ and the commutators $[d_1(u), d_2(y)] = [u, y]$ where $u \in \mathcal{N}$ and $y \in J(\mathcal{N})$, then $d_1 \neq 0$ on $J(\mathcal{N})$ and $\mathcal{N}$ is commutative.*

Proof. We show that $\mathcal{N}$ is indeed commutative iff $[u, y] = 0$, $\forall y \in J(\mathcal{N})$ whenever the Frobenius map $d_1 \neq 0$ on $J(\mathcal{N})$.

Let $u \in \mathcal{N}$ and $y \in J(\mathcal{N})$. Since the ideal $J(\mathcal{N})$ is a subset of $\mathcal{N}$, $yu \in \mathcal{N}$ and $yu \in J(\mathcal{N})$. Now, $[d_1(u), d_2(y)] = [u, y]$ for all $u \in \mathcal{N}$ and $y \in J(\mathcal{N})$. Let $x = yu$ for some $x \in \mathcal{N}$. Then $[x, y] = [yu, y] = y[u, y]$. Therefore $[d_1(yu), d_2(y)] = [yu, y] = y[u, y]$. But by definition of commutator, we see that

$$[d_1(yu), d_2(y)] = d_1(yu)d_2(y) - d_2(y)d_1(yu) = y[d_1(u), d_2(y)] = y[u, y], \ \forall y \in J(N), u \in \mathcal{N}.$$

From

$$[d_1(yu), d_2(y)] = d_1(yu)d_2(y) - d_2(y)d_1(yu), \tag{1}$$

we use d_1 and apply the definition of the multiplicative Frobenius derivation of the construction of $\mathcal{N}$ on the right hand side of equation (1) above to get,

$$[d_1(yu), d_2(y)] = \Big(yd_1(u) + d_1(y)u\Big)d_2(y) - d_2(y)\Big(yd_1(u) + d_1(y)u\Big) \tag{2}$$

$$= yd_1(u)d_2(y) + d_1(y)ud_2(y) - d_2(y)yd_1(u) - d_2(y)d_1(y)u$$
$$= yd_1(u)d_2(y) - yd_2(y)d_1(u),$$

because of the commuting property of d_2 on $J(\mathcal{N})$ and thus intuitively, equation (2) implies that,

$$d_1(y)ud_2(y) - d_2(y)d_1(y)u = \{0\} \Rightarrow d_1(y)ud_2(y) = d_2(y)d_1(y)u \tag{3}$$

Finnaly, consider some linear combination of u as $u = vm$ where $u, v, m \in \mathcal{N}$ and by quasi-3-primeness of $\mathcal{N}$ together with the condition that $d_1(y)ud_2(y) - d_2(y)d_1(y)u = \{0\}$, we see that,

$$d_1(y)vmd_2(y) - d_2(y)d_1(y)vm = \{0\} \Rightarrow d_1(y)N[d_2(y), m] = \{0\}\forall y \in J(\mathcal{N}), m \in \mathcal{N}. \tag{4}$$

Thus $d_1(y) = 0$ or $d_2(y) \in Z(\mathcal{N})$ by quasi-3 primeness of our $\mathcal{N}$. If $d_1(y) \neq 0$ then $d_2(y) \in Z(\mathcal{N})$ in which case $[u, y] = 0$ for all $y \in J(\mathcal{N})$ which implies that $y \in Z(\mathcal{N})$. We therefore conclude that $\mathcal{N}$ is commutative. $\qquad\square$

5 Conclusion

In this paper, we have introduced a new notion of Frobenius derivations on idealized quasi-3 prime zero-symmetric near-rings $\mathcal{N}$ and studied both the structural properties of $\mathcal{N}$, $J(\mathcal{N})$ and the commutation of $\mathcal{N}$ using well chosen derivations acting on $\mathcal{N}$ and $J(\mathcal{N})$. The results in the paper extend the existing studies on completely primary finite rings to ring-type near rings.

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Competing Interests

Authors have declared that no competing interests exist.

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